Global wave front sets in ultradifferentiable classes

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Outline

0. Preliminaries.
1. The ω-wave front set.
2. Global pseudodifferential operators and parametrices.
3. The Weyl wave front set.
4. Applications.

Weight functions

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(α) $\exists\, L > 1 : \omega(2t) \leq L(\omega(t)+1), \quad \forall\, t \geq 0;$

(β) $\displaystyle\int_1^\infty \frac{\omega(t)}{t^2}\,dt < +\infty;$

(γ) $\log(t) = o(\omega(t)), \quad t \to \infty;$

(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ *convex.*

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Definition (Young conjugate)

$$\varphi^*_\omega : [0, +\infty[\to [0, +\infty[, \qquad \varphi^*_\omega(t) := \sup_{s \geq 0}\{st - \varphi_\omega(s)\}.$$

Ultradifferentiable functions of Beurling type

$$\mathcal{S}(\mathbb{R}^d) := \left\{ u \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \left| x^\beta D^\alpha u(x) \right| < +\infty, \quad \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

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Definition

Let $u \in \mathcal{S}(\mathbb{R}^d)$. *Then* $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ *if* $\forall \lambda > 0 \; \exists C_\lambda > 0$:

$$\sup_{\alpha, \beta \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} |x^\beta D^\alpha u(x)| e^{-\lambda \varphi_\omega^* \left(\frac{|\alpha + \beta|}{\lambda} \right)} \leq C_\lambda.$$

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Alternatively, $\forall \lambda > 0 \, \exists C_\lambda > 0$:

$$\sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in\mathbb{R}^d} |D^\alpha u(x)| e^{-\lambda \varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)} e^{\lambda\omega(x)} \leq C_\lambda.$$

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$\mathcal{S}'_\omega(\mathbb{R}^d)$ is the dual of $\mathcal{S}_\omega(\mathbb{R}^d)$.

Time-frequency analysis

Definition (Short-time Fourier transform)

Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ *and* $0 \neq \psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ *be a window function.*

$$V_\psi u(z) := \langle u, \Pi(z)\psi \rangle = \int_{\mathbb{R}^d} e^{-it\cdot\xi}\overline{\psi(t-x)}u(t)dt,$$

for $z = (x, \xi) \in \mathbb{R}^{2d}$*, where* $\Pi(z)\psi := e^{i\cdot\xi}\psi(\cdot - x)$*.*

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for $z = (x,\xi) \in \mathbb{R}^{2d}$, *where* $\Pi(z)\psi := e^{i\cdot\xi}\psi(\cdot - x)$.

Theorem (Gröchenig, Zimmermann)

Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ *and* $0 \neq \psi \in \mathcal{S}_\omega(\mathbb{R}^d)$. *TFAE:*

1. $u \in \mathcal{S}_\omega(\mathbb{R}^d)$.
2. $\forall \lambda > 0 \; \exists C_\lambda > 0$:
$$\sup_{z\in\mathbb{R}^{2d}} e^{\lambda\omega(z)}|V_\psi u(z)| \leq C_\lambda.$$
3. $V_\psi u \in \mathcal{S}_\omega(\mathbb{R}^{2d})$.

Definition (The ω-wave front set)

Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ *and* $0 \neq \psi \in \mathcal{S}_\omega(\mathbb{R}^d)$. $z_0 \in \mathbb{R}^{2d} \setminus \{0\}$ *is not in* $\mathrm{WF}'_\omega(u)$ *if* $\exists\, \Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ *open conic set,* $z_0 \in \Gamma$*, s.t.*

$$\sup_{z\in\Gamma} e^{\lambda\omega(z)} |V_\psi u(z)| < +\infty, \qquad \lambda > 0.$$

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Definition (The ω-Gabor wave front set)

Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, $0 \neq \psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ *and* $\Lambda = \alpha_0\mathbb{Z}^d \times \beta_0\mathbb{Z}^d$ *with* $\alpha_0, \beta_0 > 0$ *small enough.* $z_0 \in \mathbb{R}^{2d} \setminus \{0\}$ *is not in* $\mathrm{WF}^G_\omega(u)$ *if* $\exists\, \Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ *open conic set,* $z_0 \in \Gamma$*, s.t.*

$$\sup_{\sigma \in \Lambda \cap \Gamma} e^{\lambda\omega(\sigma)} |V_\psi u(\sigma)| < +\infty, \qquad \lambda > 0.$$

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$$\sup_{\sigma \in \Lambda \cap \Gamma} e^{\lambda\omega(\sigma)} |V_\psi u(\sigma)| < +\infty, \qquad \lambda > 0.$$

Theorem (Boiti, Jornet, Oliaro)

$$\omega \text{ subadditive} \quad \Rightarrow \quad \mathrm{WF}'_\omega(u) = \mathrm{WF}^G_\omega(u), \qquad u \in \mathcal{S}'_\omega(\mathbb{R}^d).$$

Inclusions for LPDO

LPDO $P(x, D)$ satisfying

$$\mathrm{WF}'_\omega(P(x,D)u) \subseteq \mathrm{WF}'_\omega(u), \qquad u \in \mathcal{S}'_\omega(\mathbb{R}^d).$$

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$$P(x,D) = \sum_{|\alpha+\beta|\leq m} c_{\alpha\beta} x^\alpha D^\beta$$

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A., Boiti, Jornet, Oliaro

$$P(x,D) = \sum_{|\gamma|\leq m} a_\gamma(x) D^\gamma$$

for some $m \in \mathbb{N}$, where $a_\gamma \in \mathcal{S}_\omega(\mathbb{R}^d)$.

Pseudodifferential operators

Definition (Global symbol)

$a \in \mathrm{GS}_\rho^{m,\omega}$ ($m \in \mathbb{R}, 0 < \rho \leq 1$) *if* $a \in C^\infty(\mathbb{R}^{2d}) : \forall \lambda > 0 \; \exists C_\lambda > 0$,

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_\lambda \langle (x,\xi) \rangle^{-\rho|\alpha+\beta|} e^{\lambda \rho \varphi_\omega^*\left(\frac{|\alpha+\beta|}{\lambda}\right)} e^{m\omega(x,\xi)},$$

$\forall \alpha, \beta \in \mathbb{N}_0^d$, $x, \xi \in \mathbb{R}^d$, *where* $\langle (x,\xi) \rangle := \sqrt{1 + |x|^2 + |\xi|^2}$.

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Definition (Pseudodifferential operator)

Given $a \in \mathrm{GS}_\rho^{m,\omega}$,

$$Au(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)d\xi, \qquad u \in \mathcal{S}_\omega(\mathbb{R}^d).$$

Pseudodifferential operators II

Definition (Global amplitude)

$a \in \mathrm{GA}_\rho^{m,\omega}$ if $a \in C^\infty(\mathbb{R}^{3d}) : \forall \lambda > 0\ \exists C_\lambda > 0$,

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \leq C_\lambda \left(\frac{\langle x-y \rangle}{\langle (x,y,\xi) \rangle} \right)^{\rho|\alpha+\beta+\gamma|} e^{\lambda \rho \varphi_\omega^* \left(\frac{|\alpha+\beta+\gamma|}{\lambda} \right)} e^{m\omega(x,y,\xi)},$$

$\forall \alpha, \beta, \gamma \in \mathbb{N}_0^d$, $x, y, \xi \in \mathbb{R}^d$.

Definition (Pseudodifferential operator)

Given $a \in \mathrm{GA}_\rho^{m,\omega}$,

$$Au(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,y,\xi) u(y) dy d\xi, \quad u \in \mathcal{S}_\omega(\mathbb{R}^d).$$

Pseudodifferential operators III

Theorem

$A : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is well defined and continuous.

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Proposition

$A : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ *pseudodifferential operator. TFAE:*

1. *A has a linear and continuous extension* $\mathcal{S}'_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$.
2. $\exists K(x, y) \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ *s.t.*

$$Au(x) = \int_{\mathbb{R}^d} K(x, y)u(y)dy, \qquad u \in \mathcal{S}_\omega(\mathbb{R}^d).$$

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Definition (ω-regularizing)

$R : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ pseudodifferential operator satisfying (1).

Symbolic calculus

Definition (Formal sums)

$\sum_{j\in\mathbb{N}_0} a_j(x,\xi) \in \mathrm{FGS}_\rho^{m,\omega}$ *if* $a_j(x,\xi) \in C^\infty(\mathbb{R}^{2d})$ *and* $\exists\, r \geq 1$ *s.t.* $\forall\, n \in \mathbb{N}\ \exists\, C_n > 0$ *with*

$$\left|D_x^\alpha D_\xi^\beta a_j(x,\xi)\right| \leq C_n \frac{e^{n\rho\varphi_\omega^*\left(\frac{|\alpha+\beta|+j}{n}\right)}}{\langle(x,\xi)\rangle^{\rho(|\alpha+\beta|+j)}} e^{m\omega(x,\xi)}$$

for each $j \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{N}_0^d$ *and* $\log\left(\frac{\langle(x,\xi)\rangle}{r}\right) \geq \frac{n}{j}\varphi_\omega^*\left(\frac{j}{n}\right)$.

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for each $j \in \mathbb{N}_0$, $\alpha,\beta \in \mathbb{N}_0^d$ and $\log\left(\frac{\langle(x,\xi)\rangle}{r}\right) \geq \frac{n}{j}\varphi_\omega^*\left(\frac{j}{n}\right)$.

$\sum a_j \sim \sum b_j$ if $\exists\, r \geq 1$ s.t. $\forall\, n \in \mathbb{N}$, $\exists\, C_n > 0$, $N_n \in \mathbb{N}$ with

$$\left|D_x^\alpha D_\xi^\beta \sum_{j<N}(a_j - b_j)\right| \leq C_n \frac{e^{n\rho\varphi_\omega^*\left(\frac{|\alpha+\beta|+N}{n}\right)}}{\langle(x,\xi)\rangle^{\rho(|\alpha+\beta|+N)}} e^{m\omega(x,\xi)},$$

for every $N \geq N_n$, $\alpha,\beta \in \mathbb{N}_0^d$ and $\log\left(\frac{\langle(x,\xi)\rangle}{r}\right) \geq \frac{n}{N}\varphi_\omega^*\left(\frac{N}{n}\right)$.

Properties of formal sums

Theorem

If $\sum a_j \in \mathrm{FGS}_{\rho}^{m,\omega}$ *then* $\exists\, a \in \mathrm{GS}_{\rho}^{m,\omega}$ *s.t.* $a \sim \sum a_j$.

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Idea: Take σ a weight function s.t. $\omega\big(t^{1/\rho}\big) = o\big(\sigma(t)\big)$, $t \to \infty$.
Fix $\Phi(x,\xi) \in \mathcal{S}_\sigma\big(\mathbb{R}^{2d}\big)$ with **compact support**; set

$$\Psi_{j,n}(x,\xi) := 1 - \Phi\Big(\frac{(x,\xi)}{A_{n,j}}\Big), \qquad A_{n,j} = re^{\frac{n}{j}\varphi_\omega^*\left(\frac{j}{n}\right)},$$

and put $(j_n)_n \subset \mathbb{N}$: $j_n/n \to \infty$ and $j_n \le j < j_{n+1}$.

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and put $(j_n)_n \subset \mathbb{N}$: $j_n/n \to \infty$ and $j_n \leq j < j_{n+1}$.
Prove

$$a(x,\xi) := a_0(x,\xi) + \sum_{n=1}^{\infty} \sum_{j=j_n}^{j_{n+1}-1} \Psi_{j,n}(x,\xi) a_j(x,\xi) \in \mathrm{GS}_\rho^{m,\omega}.$$

Analysis of kernel

For any $s > 0$,

$$\Delta_s := \{(x,y) \in \mathbb{R}^{2d} : |x-y| < s\}.$$

Theorem

If $s > 0$ and $a(x,y,\xi) \in \mathrm{GA}_\rho^{m,\omega}$, then

$$K(x,y) := \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,y,\xi) d\xi$$

satisfies

1. $K(x,y) \in C^\infty(\mathbb{R}^{2d} \setminus \overline{\Delta_s})$,
2. $\forall \lambda > 0 \; \exists C_\lambda > 0$ *s.t.* $\forall \alpha, \beta \in \mathbb{N}_0^d$, $(x,y) \in \mathbb{R}^{2d} \setminus \Delta_s$,

$$|D_x^\alpha D_\xi^\beta K(x,y)| e^{-\lambda \varphi_\omega^*\left(\frac{|\alpha+\beta|}{\lambda}\right)} e^{\lambda\omega(x,y)} \leq C_\lambda.$$

Weyl quantization and composition

Theorem

For $a \in \mathrm{GA}_{\rho}^{m,\omega}$*,*

$$A = P + R,$$

where, for $u \in \mathcal{S}_{\omega}(\mathbb{R}^d)$*,*

$$Pu = p^w(x,D)u = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} p\Big(\frac{x+y}{2}, \xi\Big) u(y) dy d\xi,$$

being p *a global symbol:*

$$p(x,\xi) \sim \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j} \frac{(-1)^{|\beta|}}{\beta!\gamma!} 2^{-|\beta+\gamma|} \partial_{\xi}^{\beta+\gamma} D_x^{\beta} D_y^{\gamma} a(x,y,\xi)|_{y=x}.$$

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Corollary (Composition of pseudodifferential operators)

The Weyl symbol c_w *of* $C = A \circ B$ *is equivalent to*

$$(2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j} \frac{(-1)^{|\beta|}}{\beta!\gamma!} 2^{-|\beta+\gamma|} \partial_\xi^\gamma D_x^\beta a(x,\xi) \partial_\xi^\beta D_x^\gamma b(x,\xi).$$

Construction of parametrix

Theorem (Parametrix)

Given ω, *take* σ *a subadditive weight:* $\omega(t^{1/\rho}) = o(\sigma(t))$, $t \to \infty$.
Let $p \in \mathrm{GS}_\rho^{m,\omega}$ ($m \geq 0$) *be s.t., for some* $r \geq 1$,

(i) $\exists c > 0 : |p(z)| \geq ce^{-m\omega(z)}, \forall \langle z \rangle \geq r$;
(ii) $\exists C > 0, n \in \mathbb{N} : \forall \alpha \in \mathbb{N}_0^{2d}, \langle z \rangle \geq r$,

$$|D^\alpha p(z)| \leq C^{|\alpha|} \langle z \rangle^{-\rho|\alpha|} e^{\frac{1}{n}\varphi_\sigma^*(n|\alpha|)} |p(z)|.$$

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(ii) $\exists\, C > 0, n \in \mathbb{N} : \forall \alpha \in \mathbb{N}_0^{2d}$, $\langle z \rangle \geq r$,

$$|D^{\alpha} p(z)| \leq C^{|\alpha|} \langle z \rangle^{-\rho|\alpha|} e^{\frac{1}{n}\varphi_{\sigma}^{*}(n|\alpha|)} |p(z)|.$$

Then, $\exists\, q \in \mathrm{GS}_{\rho}^{m,\omega} : Q \circ P = I + R$.

Construction of parametrix

Theorem (Parametrix)

Given ω, *take* σ *a subadditive weight:* $\omega(t^{1/\rho}) = o(\sigma(t))$, $t \to \infty$.
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If $p \in \mathrm{GS}_\rho^{m,\omega}$ *satisfies* (i) *and* (ii),

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Twisted Laplacian

$$L = \left(D_x - \frac{1}{2}y\right)^2 + \left(D_y + \frac{1}{2}x\right)^2.$$

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Example for Gevrey weights $\omega(t) = t^a, \quad 0 < a < 1$

Take

$$g(t) = e^{mt^{a/2}}, \qquad 0 < a < 1, \;\; m \geq 0, \;\; t \geq 1;$$
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Estimate for its derivatives

$\exists\, C = 16d(1+m)(a/2+1)e > 0$ s.t.

$$|D^\alpha p(z)| \leq C^{|\alpha|}\alpha!\langle z \rangle^{-(1-a)|\alpha|}|p(z)|, \quad \alpha \in \mathbb{N}_0^{2d}, \ \ z \in \mathbb{R}^{2d}.$$

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The Weyl wave front set

Definition (non-characteristic point)

Given $a \in \mathrm{GS}_{\rho}^{m,\omega}$, $z_0 \in \mathbb{R}^{2d} \setminus \{0\}$ *is non-characteristic for a if* $\exists$ *Gevrey weight* σ *with* $\omega(t^{1/\rho}) = o(\sigma(t))$, $t \to \infty$, $C_1, C_2 > 0$, $n \in \mathbb{N}$, $r \geq 1$, *and open conic set* $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ *with* $z_0 \in \Gamma$ *s.t.*

(i') $|a(z)| \geq C_1 e^{m\omega(z)}, \ \forall z \in \Gamma, |z| \geq r$;

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Definition (conic support)

Let $u \in \mathcal{S}'_\omega(\mathbb{R}^{2d})$ *and* $z \in \mathbb{R}^{2d} \setminus \{0\}$. *We say* $z \in \operatorname{conesupp}(u)$ *if every conic set* $\Gamma \subseteq \mathbb{R}^{2d} \setminus \{0\}$ *containing z satisfies*

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For $a \in \mathrm{GS}_{\rho}^{m,\omega}$, $\ \mathbb{R}^{2d} \setminus \{0\} = \operatorname{conesupp}(a) \cup \operatorname{char}(a)$.

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Let $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $0 < \rho \leq 1$. $z_0 \in \mathbb{R}^{2d} \setminus \{0\}$ is not in $\mathrm{WF}^\omega_\rho(u)$ if $\exists\, m \in \mathbb{R}$ and $a \in \mathrm{GS}^{m,\omega}_\rho$ s.t. $a^w(x, D)u \in \mathcal{S}_\omega(\mathbb{R}^d)$ and z_0 is non-characteristic for a.

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ω is ρ-regular if $\forall\, m \in \mathbb{R}\ \exists\, a \in \mathrm{GS}^{m,\omega}_\rho$ s.t. for some Gevrey weight σ with $\omega(t^{1/\rho}) = o(\sigma(t)), t \to \infty$, $C_1, C_2 > 0$, $n \in \mathbb{N}$,

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1. $\omega(t) = t^a$, $0 < a < 1/2$, is $(1-a)$-regular.
2. $\omega(t) = \log(1+t)$ is ρ-regular, $0 < \rho \leq 1$.

Equality of wave front sets

First inclusion

Let ω be a ρ-regular weight function, for some $0 < \rho \leq 1$. Then,

$$\mathrm{WF}'_\omega(u) \subseteq \mathrm{WF}^\omega_\rho(u), \qquad u \in \mathcal{S}'_\omega(\mathbb{R}^d).$$

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Second inclusion

Let ω be a weight function. If for some $0 < \rho \leq 1$,

$$\omega(t^{1/\rho}) = o(\sigma(t)) \quad \text{and} \quad \sigma(t^{1+\rho/2}) = O(\gamma(t)), \;\; t \to \infty, \tag{$*$}$$

for some Gevrey weight σ and some weight γ, then,

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Weight functions satisfying the equality of WFS

Consider $\omega(t) = t^a$, where $0 < a < \frac{5-\sqrt{17}}{2} \left(\lneq \frac{1}{2} \right)$.
Enough: $\sigma(t) = t^b$ and $\gamma(t) = t^c$, where

$$\frac{a}{1-a} < b < \frac{2}{3-a} \qquad\qquad b\left(\frac{3-a}{2}\right) < c < 1.$$

Regularity of Weyl quantizations

Theorem

Let ω be a ρ-regular weight function, $a \in \mathrm{GS}_{\rho}^{m,\omega}$. Then,

$$\begin{aligned}\mathrm{WF}_{\rho}^{\omega}(a^{w}(x,D)u) &\subseteq \mathrm{WF}_{\rho}^{\omega}(u) \cap \mathrm{conesupp}\,(a)\\ &\subseteq \mathrm{WF}_{\rho}^{\omega}(u) \subseteq \mathrm{WF}_{\rho}^{\omega}(a^{w}(x,D)u) \cup \mathrm{char}\,(a),\end{aligned}$$

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$\forall\, u \in \mathcal{S}'_{\omega}(\mathbb{R}^d)$.

Corollary

Let ω be a ρ-regular weight function with $(*)$, $a \in \mathrm{GS}^{m,\omega}_{\rho}$. Then,

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Publications

V. Asensio, C. Boiti, D. Jornet, and A. Oliaro.

Global wave front sets in ultradifferentiable classes.
Results. Math., **77** *(2022), no. 2, Paper No. 65, 40 pp.*

V. Asensio.

Quantizations and global hypoellipticity for pseudodifferential operators of infinite order in classes of ultradifferentiable functions.
Mediterr. J. Math., **19** *(2022), Paper No. 135, 36 pp.*

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