q-Nagumo norms and formal solutions of singularly perturbed q-difference equations

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Introduction

In the study of analytic problems at regular points, analytic solutions (convergent power series) generically exist. It is at singular points where formal (divergent) solutions appear.

Example (Irregular singularities of ODEs)

The equation

$$x^{p+1}\frac{\partial y}{\partial x}(x) = F(x,y) = c(x) + A(x)y + \cdots$$

admits a unique formal power series solution $\hat{y} = \sum_{n=0}^{\infty} a_n x^n$, when $A(0)$ is invertible. It generically holds that

$$|a_n| \leq CA^n n!^{1/p}.$$

Example (Singularly perturbed problems)

The doubly singular equation

$$\epsilon^\sigma x^{p+1}\frac{\partial y}{\partial x}(x,\epsilon)=F(x,\epsilon,y)=c(x,\epsilon)+A(x,\epsilon)y+\cdots,$$

as before has a unique formal solution

$$\hat{y}(x,\epsilon)=\sum_{n,m=0}^{\infty}a_{n,m}x^n\epsilon^m,\qquad |a_{n,m}|\leq CA^{n+m}\min\{n!^{1/p},m!^{1/\sigma}\}.$$

Example (PDEs with normal crossings)

$$L_\lambda:=\lambda_1x_1\partial_{x_1}+\cdots+\lambda_dx_d\partial_{x_d},\qquad x_1^{\alpha_1}\cdots x_d^{\alpha_d}L_\lambda(y)=F(x,y),$$

$$|a_\beta|\leq CA^{|\beta|}\min\{\beta_1!^{1/\alpha_1},\ldots,\beta_d!^{1/\alpha_d}\}.$$

The correct source of divergence

In these type of problems a correct choice of the main variable gives the source of divergence of the solution.

In general, working in $(\mathbb{C}^d, 0)$ with coordinates $(x_1, \ldots, x_d)$, fix a germ $P \in \mathbb{C}\{x\}$ with $P(0) = 0$.

Roughly speaking, $\hat{f} \in \mathbb{C}[[x]]$ is a P-s–*Gevrey series* if we can write

$$\hat{f} = \sum_{n=0}^{\infty} f_n(x) P(x)^n, \qquad \text{where } \sup_{x \in D} |f_n(x)| \leq C A^n n!^s.$$

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For the case $P(x) = x^\alpha$ this precisely means that

$$|a_\beta| \leq C A^{|\beta|} \min\{\beta_1!^{s/\alpha_1}, \ldots, \beta_d!^{s/\alpha_d}\}.$$

- J. Mozo-Fernández and R. Schäfke, Asymptotic expansions and summability with respect to an analytic germ, Publ. Mat., **63**, 3–79 (2019).

Families of singular PDEs

Theorem (2020)

Consider the analytic PDE

$$P(x)L(y)(x) = P(x)\left(a_1(x)\frac{\partial y}{\partial x_1} + \cdots + a_d(x)\frac{\partial y}{\partial x_d}\right) = F(x,y),$$

where $x \in (\mathbb{C}^d, 0)$*,* $y \in \mathbb{C}^N$*,* $F(0,0) = 0$ *and* $D_yF(0,0)$ *is invertible. If*

$$P \text{ divides } L(P),$$

the equation has a unique solution $\widehat{y} \in \mathbb{C}[[x]]^N$ *which is* 1*–Gevrey in* P*.*

- S.A. Carrillo, A. Lastra. Formal Gevrey solutions - in analytic germs - for higher order holomorphic PDEs. Math. Ann. (2022) doi 10.1007/s00208-022-02393-w.

Singularly perturbed and doubly singular ODEs

Consider

$$Q(\epsilon)x^{k+1}\frac{\partial y}{\partial x} = F(x, \epsilon, y).$$

- For $k = -1$, $Q(0) = 0$, choose

$$P = Q(\epsilon), \quad L = \partial_x, \quad L(P) = 0, \quad Q(\epsilon)\text{-1-Gevrey solution.}$$

- For $k \geq 0$, take

$$P = x^k Q(\epsilon), \quad L = x\partial_x, \quad L(P) = kx^k Q(\epsilon), \quad x^k Q(\epsilon)\text{-1-Gevrey solution.}$$

This recovers the well-known case $Q(\epsilon) = \epsilon^q$.

- Canalis-Durand M., Ramis J.P., Schäfke R., Sibuya Y. *Gevrey solutions of singularly perturbed differential equations.* J. Reine Angew. Math, vol. 518, (2000) 95–129.
- Canalis-Durand M., Mozo-Fernández J., Schäfke R. *Monomial summability and doubly singular differential equations.* J. Differential Equations, vol. 233, (2007) 485–511.

A simple strategy: lifting the dimension by one

Set $\widehat{W}(x,t) = \sum_{n=1}^{\infty} y_n t^n$, where $\widehat{y}(x) = \sum_{n=1}^{\infty} y_n P^n$, i.e,

$$\widehat{y}(x) = \widehat{W}(x, P(x)).$$

We can assume the equation takes the form

$$P \cdot L(y) = g_0(x) + B_0(x)y + H_0(x,y), \tag{1}$$

where $g_0 = P \cdot h_0$ and $B_0(x)$ is invertible at $x = 0$. Then, observe that

$$P \cdot L(\widehat{y}) = \sum_{n=1}^{\infty} L(y_n)P^{n+1} + \phi \cdot n y_n P^{n+1} = \left(tL + \phi t^2 \partial_t\right)(\widehat{W})\Big|_{t=P},$$

where $L(P) = \phi \cdot P$. Therefore, $\widehat{y}$ solves (1) if and only $\widehat{W}$ solves

$$B_0(x)W = -h_0 \cdot t + \left(tL + \phi(x)t^2\partial_t\right) W - H_0(x,W). \tag{2}$$

Classical theorems shows (2) has a unique solution $\widehat{W}(x,t)$ which is 1-Gevrey.

- Gérard R., Tahara H.: Singular nonlinear partial differential equations. Aspects of Mathematics. E28. Wiesbaden: Vieweg. viii (1996).

The Nagumo norms

The classical Nagumo norms are defined for $f \in \mathcal{O}(D_r)$ and $m \in \mathbb{N}$ as

$$\|f\|_m := \sup_{|x|<r} |f(x)|(r-|x|)^m.$$

These are useful to establish convergence/Gevrey type of solutions of differential equations and singularly perturbed problems.

I. $\|f+g\|_m \leq \|f\|_m + \|g\|_m$ and $\|fg\|_{m+k} \leq \|f\|_m \|g\|_k$.

II. $\|f'\|_{m+1} \leq e(m+1)\|f\|_m$.

- Nagumo M. *Über das anfangswertproblem partieller differentialgleichunge*, Jap. J. Math. 18 (1942), 41–47.
- Canalis-Durand M., Ramis J.P., Schäfke R., Sibuya Y. *Gevrey solutions of singularly perturbed differential equations.* J. Reine Angew. Math, vol. 518, (2000) 95–129.

An example from analytic flows

Let $X = (X_1, \ldots, X_n) \in \mathbb{C}\{z\}^n$ be a vector field in $(\mathbb{C}^n, 0)$. The dynamics of X is determined through its flow $\phi_X(t, z)$, which is the solution of the differential equation

$$\partial_t \phi_X(t, z) = X(\phi_X(t, z)), \qquad \phi_X(0, z) = z. \tag{3}$$

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Consider the auxiliary problem

$$\partial_t w(t, z) = D_z w(t, z) \cdot X(z), \qquad w(0, z) = z.$$

Setting

$$w(t, z) = \sum_{m=0}^{\infty} \varphi_m(z) \frac{t^m}{m!},$$

we find $\varphi_0(z) = w(0, z) = z, \quad \varphi_{m+1}(z) = D_z \varphi_m(z) \cdot X(z), \quad m \geq 0.$
Therefore $\varphi_m(z) = X^m(\mathrm{id})(z) = (X^m(z_1), \dots, X^m(z_n))$.

Considering

$$W(\tau) := \sum_{m=0}^{\infty} \|\varphi_m\|_m \frac{\tau^m}{m!},$$

we see that $\dfrac{\|\varphi_{m+1}\|_{m+1}}{(m+1)!} \leq \dfrac{\|D_z\varphi_m\|_{m+1}}{(m+1)!}\|X\|_0 \leq ner^{n-1}\|X\|_0 \dfrac{\|\varphi_m\|_m}{m!}$. Thus

$$\frac{\|\varphi_m\|_m}{m!} \leq \|\varphi_0\|_0\alpha^m, \qquad \alpha := ner^{n-1}\|X\|_0.$$

Theorem

The problem (3) admits a unique analytic solution $\phi_X(t,z) \in \mathbb{C}\{t,z\}^n$ which is given by the Lie series

$$\phi_X(t,z) = \sum_{m=0}^{\infty} (X^m(z_1), \ldots, X^m(z_n)) \frac{t^m}{m!}.$$

- Carrillo S. A. *A quick proof of the regularity of the flow of analytic vector fields*, C. R. Mathématique (2021) 359(9), 1155-1159.

Interlude on M-sequences

Let $M = (M_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. We consider the following conditions:

1. (Log-convexity) $M_n^2 \leq M_{n-1}M_n$ for all $n \in \mathbb{N}_{>0}$, i.e., $\boldsymbol{m}$ is a non-decreasing sequence.
2. (Moderate growth) $M_{n+m} \leq A^{n+m}M_nM_m$ for all $n, m \in \mathbb{N}$, for some constant $A > 0$.

Recall that moderate growth implies the existence of $\delta, A > 0$ with

$$M_n \leq A^n n!^\delta.$$

M-series in a monomial

The natural extension would be to say that $\hat{f}$ is a $M-$*series in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ if there is $r>0$ and $B,D>0$ such that

$$\|f_{\boldsymbol{\alpha},n}\| \leq DB^n M_n, \qquad n \in \mathbb{N}.$$

Assuming that M has moderate growth, this is equivalent to the existence of $C,B>0$ satisfying

$$|a_{\boldsymbol{\beta}}| \leq CB^{|\boldsymbol{\beta}|} \min(M_{\beta_1}^{1/\alpha_1},\ldots,M_{\beta_d}^{1/\alpha_d}), \quad \boldsymbol{\beta} \in \mathbb{N}^d.$$

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What happens if not? For instance,

$$M_n = q^{n^2/2}, \qquad q^{n(n-1)/2}, \qquad q^{n(n+1)/2}.$$

Review on q-calculus

This type of Calculus replaces the usual derivative by the discrete analogue

$$d_q(f)(x) := \frac{f(qx) - f(x)}{qx - x}, \qquad q \in \mathbb{C} \setminus \{0, 1\}.$$

In this framework we have q-analogues to classical coefficients, functions and analytic equations.

These are expected to be *confluent* to the usual versions as $q \to 1$.

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Another operator that is more common in the literature is

$$\sigma_q(y)(x) := y(qx).$$

- J. Cano, P. Fortuny-Ayuso, Power series solutions of non-linear q-difference equations and the Newton-Puiseux Polygon (2012) arxiv.org/abs/1209.0295.
- C. Zhang, Sur un théorème du type de Maillet-Malgrange pour les équations q-différences-différentielles, *Asymptot. Anal.* 17, no. 4, 309–314 (1998).
- L. Di Vizio, Ch. Zhang, On $q-$summation and confluence, *Ann. Inst. Fourier* 59, No. 1, 347–392 (2009).

The q-Taylor formula

If $n \in \mathbb{N}$, then

$$d_q(x^n) = \frac{(qx)^n - x^n}{(q-1)x} = [n]_q x^{n-1}, \quad [n]_q := \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

Therefore, for any formal power series $f(x) \in \mathbb{C}[[x]]$ and $|q| \neq 0, 1$ we have that

$$f(x) = \sum_{j=0}^{\infty} \frac{d_q^j(f)(0)}{[j]_q^!} x^j, \qquad [n]!_q := [1]_q [2]_q \cdots [n]_q.$$

The non-singular case

Theorem

Let $q > 0$ where $q \neq 1$. The problem

$$d_q(y)(x) = a(x)y(x) + b(x)y(qx) + c(x), \qquad y(0) = y_0, \tag{4}$$

where $a, b, c \in \mathbb{C}\{x\}$ has a unique analytic solution $\hat{y} \in \mathbb{C}\{x\}$.

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Example (q-exponential maps)

$$d_q(y)(x) = y(x), \qquad y(0) = 1, \qquad e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]!_q},$$

$$d_q(y)(x) = y(qx), \qquad y(0) = 1, \qquad E_q(x) := \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]!_q}.$$

q-Euler's equation

Consider the q-analogue to Euler's equation

$$x^2 d_q y(x) + y(x) = x.$$

It has a unique formal solution

$$\hat{E}_q(x) := \sum_{n=0}^{\infty} (-1)^n [n]_q^! x^{n+1},$$

which is divergent for $q > 1$. In fact, note that

$$n \leq [n]_q \leq nq^{n-1}, \qquad n! \leq [n]_q! \leq n! q^{n(n-1)/2}.$$

Letting $q \to 1^+$, $\hat{E}_q(x) \to \hat{E}(x) := \sum_{n=0}^{\infty} (-1)^n n^! x^{n+1}$.

Another q-difference equation

Consider the problem

$$x\sigma_q(y)(x) = y(x) - 1.$$

It has the unique formal power series solution

$$\sum_{n=0}^{\infty} q^{n(n-1)/2} x^n.$$

In terms of d_q we have the problem

$$(q-1)x^2 d_q(y)(x) = (1-x)y(x) - 1.$$

In the limit $q \to 1^+$, the solution becomes the geometric series.

q-Gevrey series

Let $s \geq 0$ and $q > 1$. A series $\hat{f} = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[[x]]$ is q-s-*Gevrey* if there are $B, D > 0$ with

$$|a_n| \leq DB^n \left(q^{n^2/2}\right)^s, \qquad n \in \mathbb{N}.$$

We have the limit

$$\lim_{n \to +\infty} [n]!_q / \frac{q^{\frac{n(n-1)}{2}}}{(1-1/q)^n} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{q^k}\right).$$

For a fixed $q > 1$, up to a change in the constants, we request that

$$|a_n| \leq CA^n ([n]_q!)^s, \qquad n \in \mathbb{N}.$$

The q-analogues to the initial problems

Fix $q > 1$. We consider the q-analogues

$$\epsilon x^2 d_{q,x} y(x,\epsilon) = c(x,\epsilon) + A(x,\epsilon) y + \cdots .$$

The q-analogues to the initial problems

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What is the type of divergence of the unique formal power series solution of each one of them?
In the variable x (x_1) we have a similar behavior as in the differential case:

$$|a_{n,m}| \leq CA^{n+m}[n]_q!.$$

The type of series involved

Writing $\hat{y} = \sum_{m=0}^{\infty} u_m(x)\epsilon^m$, if we start with

$$u_0(x) \in \mathcal{O}_b(D_r), \qquad \text{then} \qquad u_m(x) \in \mathcal{O}_b(D_{r/q^m}).$$

In fact,

$$x^2 d_q(u_{m-1})(x) = c_{*m}(x) + A_{*0}(x)u_m(x) + \sum_{j=0}^{m-1} A_{*m-j}(x)u_j(x) + \cdots .$$

The q-Nagumo norms

Motivated by the nature of the problem, we consider the following variation: for $q > 1$, $n \in \mathbb{N}$, and $f \in \mathcal{O}(D_{r/q^n})$, let

$$\|f\|_n := \sup_{|x|<r/q^n} |f(x)|(r - q^n|x|)^n.$$

In this case,

I. $\|f + g\|_n \leq \|f\|_n + \|g\|_n$ and $\|fg\|_{n+m} \leq \|f\|_n\|g\|_m$.

II. $\|d_q(f)\|_{n+1} \leq eq^n(n+1)\|f\|_n$.

III. $\|\sigma_q(f)\|_{n+1} \leq r\|f\|_n$.

For the equation $\epsilon x^2 d_{q,x} y(x,\epsilon) = c(x,\epsilon) + A(x,\epsilon)y + \cdots$, we find using these norms that

$$\|u_m\|_m \leq R^m m! q^{m(m-1)/2},$$

and taking into account the restriction on the radius, we see that

$$|a_{n,m}| \leq K^{n+m} q^{nm} m! q^{m(m-1)/2}.$$

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In conclusion, the solution exhibits a divergence of type

$$|a_{n,m}| \leq K^{n+m} \min\{[n]_q!, q^{nm} m! q^{m(m-1)/2}\}.$$

Note also the confluence to the usual case as $q \to 1^+$.

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Note also the confluence to the usual case as $q \to 1^+$.
More generally,

$$|a_{n,m}| \leq CA^{n+m} \min\{[n]!_q^{1/p}, q^{nm/\sigma} q^{m^2/(2\sigma^2)} m!^{1/\sigma}\}.$$

Lifting again

Consider now the q-difference equation

$$x_1x_2\left(\lambda_1 x_1 d_{q,x_1} y + \lambda_2 x_2 d_{q,x_2} y\right) = c(x_1,x_2) + A(x_1,x_2)y + \cdots .$$

In this case, for $P = x_1x_2$, $\widehat{W}(x,t) = \sum_{n=1}^{\infty} y_n t^n$, and $\widehat{y}(x) = \sum_{n=1}^{\infty} y_n P^n$. Therefore,

$$P \cdot L(\widehat{y}) = \sum_{n=1}^{\infty} L(y_n)P^{n+1} + [n]_q\left(\lambda_1 y_n(qx_1,x_2) + \lambda_2 y_n(x_1,qx_2)\right)P^{n+1}.$$

The lifted equation is

$$\left(tL + (\lambda_1\sigma_{q,x_1} + \lambda_2\sigma_{q,x_2})t^2 d_{q,t}\right)(w) = \cdots .$$

It turns out, using again the q-Nagumo norms that

$$|a_{n,m}| \leq C^{n+m} q^{\min(n,m)|m-n|} \min\{n! q^{n(n-1)/2}, m! q^{m(m-1)/2}\}.$$

This is equivalent to write

$$\hat{y} = \sum_{n=0}^{\infty} y_n(x_2) x_1^n = \sum_{m=0}^{\infty} u_m(x_1) x_2^m$$

and obtain bounds of type

$$\sup_{|x_2|<\frac{r}{q^n}} |y_n(x_2)| \leq K^n n! q^{n(n-1)/2}, \qquad \sup_{|x_1|<\frac{r}{q^m}} |u_m(x_1)| \leq K^m m! q^{m(m-1)/2}.$$

An explicit example?

Consider the scalar equation

$$\varepsilon x^2 d_q(y) = (1+x)y - x\varepsilon.$$

$$\hat{y}(x,\varepsilon) = \sum_{n=1}^{\infty} y_n(\varepsilon)x^n = \sum_{m=1}^{\infty} u_m(x)\varepsilon^m, \qquad y_n(x) = \varepsilon \prod_{j=1}^{n-1}([j]_q\varepsilon - 1).$$

We find that

$$u_1(x) = \frac{x}{1+x}, \qquad u_m(x) = \frac{x^2}{1+x} d_q(u_{m-1})(x), \quad m \geq 2.$$

Therefore,

$$u_m(x) = \frac{x^m}{(1+x)^m(1+qx)^{m-1}(1+q^2x)^{m-2}\cdots(1+q^{m-1}x)} P_m(x,q),$$

where $P_m(x,q) \in \mathbb{C}[x,q]$.

$$
\begin{aligned}
P_5(x,q) = &- q^{16}x^9 - x^8\left(-6q^{15} - 7q^{14} - 3q^{13}\right) \\
&- x^7\left(-10q^{15} - 19q^{14} - 14q^{13} - 8q^{12} - 5q^{11} - 3q^{10}\right) \\
&- x^6\left(-5q^{15} - 11q^{14} - 4q^{13} + 7q^{12} + 12q^{11} + 8q^{10} + 6q^9 + q^8\right) \\
&- x^5\left(-q^{15} - 2q^{14} + 8q^{13} + 35q^{12} + 64q^{11} + 71q^{10} + 61q^9 + 40q^8 + 19q^7 + \right. \\
&- x^4\left(3q^{13} + 22q^{12} + 55q^{11} + 84q^{10} + 98q^9 + 93q^8 + 69q^7 + 37q^6 + 12q^5 + \right. \\
&- x^3\left(4q^{12} + 15q^{11} + 30q^{10} + 44q^9 + 54q^8 + 50q^7 + 34q^6 + 18q^5 + 8q^4 + 2\right. \\
&- x^2\left(q^{11} + 3q^{10} + 5q^9 + 5q^8 - 10q^6 - 15q^5 - 13q^4 - 8q^3 - 2q^2\right) \\
&- x\left(-q^8 - 4q^7 - 11q^6 - 18q^5 - 21q^4 - 18q^3 - 10q^2 - 3q\right) \\
&+ q^6 + 3q^5 + 5q^4 + 6q^3 + 5q^2 + 3q + 1,
\end{aligned}
$$

$$\vdots$$

Thanks for your attention.