Dynamics and spectra of composition operators on the Schwartz space

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Joint work with
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Composition operators: $C_\varphi(f) = f \circ \varphi$

- Broadly studied in spaces of analytic functions on the unit disc.
- Bonet-Domanski: Dynamics and spectra of composition operators on the space of real analytic functions (2011, 2012, 2017).
- Kennesey-Wengenroth (2011), Przestacki (204, 2017), Golinski-Przestacki (2020): composition operators on $C^\infty(\mathbb{R})$.
- More recent work by Albanese-Jordá-Mele (2022) and Debrouwere-Nyet (2022).

Let X be a Fréchet space.

An operator T on X is said to be hypercyclic if there exist a dense orbit $O(T,x) := \{T^n(x) :\ n \in \mathbb{N}\}$.
The operator is supercyclic if there exists $x \in X$ such that the projective orbit $\mathbb{K}O(T,x) = \{\lambda T^n(x) :\ \lambda \in \mathbb{K},\ n \in \mathbb{N}\}$ is dense.

$T : X \to X$ is said to be power bounded if $\{T^n : n \in \mathbb{N}\}$ is an equicontinuous set. By Banach Steinhaus principle, T is power bounded if and only if $\{T^n(x) :\ n \in \mathbb{N}\}$ is bounded for each $x \in X$.

Given $T \in L(X)$, the Cesàro mean of T is defined as $T_{[n]} = \sum_{k=1}^{n} T^k/n$. T is said to be mean ergodic when $T_{[n]}$ converges to an operator P, in the strong operator topology, i.e. if $(T_{[n]}(x))$ is convergent to $P(x)$ for each $x \in X$. The operator is called *uniformly mean ergodic* if this convergence happens uniformly on bounded sets.

If X is reflexive, power boundedness $\Rightarrow$ mean ergodicity (Lorch for Banach spaces), (Albanese-Bonet-Ricker for Fréchet spaces).
For Fréchet-Montel spaces, mean ergodicity and uniform mean ergodicity are equivalent concepts.

The Schwartz space

Given $f : \mathbb{R} \to \mathbb{C}$, $f \in \mathcal{S}(\mathbb{R})$ if $f \in C^\infty(\mathbb{R})$ and

$$\pi_n(f) := \sup_{x\in\mathbb{R}} \sup_{1\leqslant j\leqslant n} (1 + |x|^2)^n |f^{(j)}(x)| < \infty$$

for each $n \in \mathbb{N}$.

$\mathcal{S}(\mathbb{R})$ is a Fréchet-Montel space when endowed with the topology generated by the sequence of seminorms $(\pi_n)_{n\in\mathbb{N}}$.

The symbols for $\mathcal{S}(\mathbb{R})$

A function $\varphi : \mathbb{R} \to \mathbb{R}$ is a symbol of $\mathcal{S}(\mathbb{R})$ if the composition operator $C_\varphi(f) = f \circ \varphi$ maps $\mathcal{S}(\mathbb{R})$ continuously into itself.

Theorem (Galbis-Jordá)

A function $\varphi \in C^\infty(\mathbb{R})$ is a symbol for $\mathcal{S}(\mathbb{R})$ if and only if the following conditions are satisfied:

(i) For all $j \in \mathbb{N}_0$ there exist $C, p > 0$ such that

$$|\varphi^{(j)}(x)| \leqslant C(1 + |\varphi(x)|^2)^p$$

for every $x \in \mathbb{R}$.

(ii) There exists $k > 0$ such that $|\varphi(x)| \geqslant |x|^{1/k}$ for all $|x| \geqslant k$.

Composition operators on $\mathcal{S}(\mathbb{R})$ are never hyperciclic: No orbit can be dense since all the functions in the orbit

$$O(C_\varphi, f) = \{f \circ \varphi_n : \; n \in \mathbb{N}\}$$

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Even more, $C_\varphi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is not supercyclic.

Proposition

For a symbol φ the composition operators C_φ is power bounded if and only if the following statements hold

(i) For all $j \in \mathbb{N}_0$ there exist $C, p > 0$ such that

$$\left|(\varphi_n)^{(j)}(x)\right| \leqslant C(1 + \varphi_n(x)^2)^p$$

for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$.

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Theorem

Let φ be a monotonic symbol

(i) If φ is increasing then C_φ is power bounded if and only if $\varphi(x) = x$ for each $x \in \mathbb{R}$. Moreover, if φ has some fixed point, C_φ is mean ergodic if and only if it is power bounded.

(ii) If φ is decreasing then C_φ is power bounded if and only if C_φ is mean ergodic if and only if $\varphi_2(x) = x$ for each $x \in \mathbb{R}$.

Polynomials of degree 1

Corollary

$\varphi(x) = ax + b$ has degree one then C_φ is mean ergodic if and only if $\varphi(x) = x$ or $\varphi(x) = -x + b$.

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Proof.

For $a > 0$. If $a \neq 1$, φ has a fixed point, hence it is mean ergodic if and only if it is power bounded which cannot be.

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Proof.

For $a > 0$. If $a \neq 1$, φ has a fixed point, hence it is mean ergodic if and only if it is power bounded which cannot be.When $a = 1$, that is $\varphi(x) = x + b$. If C_φ is mean ergodic, then

$$\frac{1}{n} C_{\varphi_n} f \to 0$$

for every f. Take $f \in \mathcal{S}(\mathbb{R})$ with $f(0) = 1$. Then, as $\varphi_n(-nb) = 0$, we must have

$$\frac{1 + n^2 b^2}{n} = \frac{1 + n^2 b^2}{n} C_{\varphi_n} f(-nb) \to_n 0$$

which implies $b = 0$.

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If $a < 0$, since φ_2 has to be the identity, then $\varphi(x) = -x + b$. $\square$

Power boundedness of polynomials of higher degree

Theorem

Let φ be a polynomial with degree greater than one. Then, the following are equivalent:

(i) C_φ is power bounded.

(ii) C_φ is mean ergodic.

(iii) φ has no fixed points ($\Rightarrow$ φ has even degree).

Sketch of the proof: (iii)$\Rightarrow$ (i)

After replacing φ by some of its iterates we can assume that φ has neither zeros nor fixed points and, for every $K > 0$ there is $m_K \in \mathbb{N}$ such that

$$|\varphi_{m+1}(t)| \geqslant K\,(\varphi_m(t))^2 \quad \forall m \geqslant m_K, \quad \forall t \in \mathbb{R}.$$

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As φ is a polynomial which does not vanish, there is $C \geqslant 1$ such that $|\varphi^{(j)}(t)| \leqslant C|\varphi(t)|$ for every $t \in \mathbb{R}$ and $j \in \mathbb{N}$.
Let us write

$$C_n = C \sum \frac{1}{k_1! \ldots k_n!}$$

where the sum is extended to all multi-indices such that

$$k_1 + 2k_2 + \ldots + nk_n = n.$$

We may find an increasing sequence (m_n) of natural numbers with the property that

$$\varphi_{m+1}(t) \geqslant C_n\,(\varphi_m(t))^2 \quad \forall m \geqslant m_n, \quad \forall t \in \mathbb{R}.$$

Proof continues

We **claim** that there is $B_n > C_n$ such that for $m \geqslant m_n$, and $n \in \mathbb{N}$,

$$\left|\frac{(\varphi_m)^{(n)}(t)}{n!}\right| \leqslant B_n^n \left|\varphi_m(t)\right|^{2n} \tag{1}$$

for every $t \in \mathbb{R}$.

Proof continues

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Let us check this for $n = 1$. We take $B_1 > C$ such that the previous inequality is satisfied for $m = m_1$. Now, assuming that the inequality holds for $n = 1$ and some $m \geqslant m_1$ we obtain

$$\begin{aligned}\left|\frac{(\varphi_{m+1})'(t)}{(\varphi_{m+1}(t))^2}\right| &= \frac{|\varphi'(\varphi_m(t)) \cdot (\varphi_m)'(t)|}{|\varphi_{m+1}(t)|^2} \leqslant \frac{C|\varphi_{m+1}(t)|B_1|\varphi_m(t)|^2}{|\varphi_{m+1}(t)|^2} \\ &= CB_1 \frac{|\varphi_m(t)|^2}{|\varphi_{m+1}(t)|} \leqslant B_1.\end{aligned}$$

Consequently (1) holds for $n = 1$ and $m \geqslant m_1$.

Spectrum of C_φ for polynomial symbols of degree 1

$\sigma(C_\varphi)$ is the spectrum of C_φ, that is the set of all complex numbers λ such that $C_\varphi - \lambda I : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ does not admit a continuous linear inverse.

Two polynomials φ, ψ are linearly equivalent if there exists $\ell(x) = xc + d$ ($c \neq 0$) such that $\psi = \ell^{-1} \circ \varphi \circ \ell$ which implies that $\sigma(C_\psi) = \sigma(C_\varphi)$.

Proposition

Given $\varphi(x) = ax + b$ with $a \neq 0$. Then

(i) If $|a| \neq 1$, $\sigma(C_\varphi) = \mathbb{C} \setminus \{0\}$.

(ii) If $a = 1$ and $b \neq 0$, $\sigma(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

(iii) If $a = -1$, then $\sigma(C_\varphi) = \{-1, 1\}$.

Proof.

If $|a| \neq 1$, φ is linearly equivalent to $\psi(x) = ax$ and it can be proved that $\sigma(\psi) = \mathbb{C} \setminus \{0\}$.
For $a = 1$, φ is linearly equivalent to $\psi(x) = x + 1$ and one can show that $\sigma(\psi) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.
Finally, if $a = -1$, $\varphi \circ \varphi(x) = x$, for all x. □

The spectrum for $\varphi(x) = x + 1$

C_φ has no eigenvalues:
$C_\varphi(f) = \lambda f$, $\lambda \neq 0$, $f \neq 0$, $\Rightarrow f(x+n) = \lambda^n f(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ hence $|\lambda| < 1$, but then, $f(x) = \lambda^n f(x-n)$ forces $f \equiv 0$.

$C_\varphi - \lambda I$ is surjective for $|\lambda| \neq 1$: It is easy to see that $\forall \ell, n$

$$\pi_\ell(C_{\varphi_n} f) \leqslant (1+4n^2)^\ell \pi_\ell(f)$$

from where the convergence of

$$g = -\sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} C_\varphi(f)$$

for $|\lambda| > 1$ follows and $C_\varphi g - \lambda g = f$. For $0 < |\lambda| < 1$ we may argue with $\psi(x) = x - 1$.

Proof continues

For $\lambda = e^{2\pi i\omega}$, $\omega \in \mathbb{R}$, if $C_\varphi(f) = \lambda f + g$, iterating we get

$$f(x+n) = \lambda^n f(x) + \sum_{k=0}^{n-1} \lambda^{n-k-1} g(x+k) \text{ and therefore } f(x) = \lambda^n f(x-n) + \sum_{k=1}^{n} \lambda^{k-1} g(x-k),$$

hence

$$f(x) = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \lambda^{-k} g(x+k) \text{ and } f(x) = \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda^{k} g(x-k).$$

Then, the Zack transform of g satisfies

$$Zg(x,\omega) = \sum_{k\in\mathbb{Z}} \lambda^k g(x-k) = 0 \; \forall x \in \mathbb{R}.$$

Consequently

$$\widehat{g}(\omega) = \int_0^1 Zg(x,\omega)dx = 0,$$

which means that the range of $C\varphi - \lambda$ cannot be surjective.

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Let φ be a polynomial of degree greater than 1. C_φ is mean ergodic if and only if $\sigma(C_\varphi) = \{0\}$.

Theorem

Let φ be a polynomial of degree greater than 1 and with fixed points. Then $\sigma(C_\varphi) \supset \overline{\mathbb{D}} \setminus \{0\}$.

Spectrum of quadratic polynomials

A quadratic polinomial $\varphi(x) = a_0 + a_1 x + a_2 x^2$ ($a_2 \neq 0$) is linearly equivalent to $\psi(x) = x^2 + c$ where $c = a_0 a_2 + \frac{a_1}{2} - \frac{a_1^2}{4}$.

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$c = \frac{1}{4}$ is the case where φ has a single fixed point of multiplicity 2. Spectrum?

Spectrum of C_φ for $\varphi(x) = x^2 + \frac{1}{4}$

Theorem

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Let $\varphi(x) = x^2 + \frac{1}{4}$ be given. Then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.

Proof.

Since φ has a fixed point, $\sigma(C_\varphi) \supset \overline{\mathbb{D}} \setminus \{0\}$. But $0 \in \sigma(C_\varphi)$ because the range of φ is $[\frac{1}{4}, \infty)$.
We have to show that $C_\varphi - \lambda I$ is a bijection for every $|\lambda| > 1$.
Injectivity:
$\overline{C_\varphi(f) = \lambda f} \Rightarrow f(\varphi_n(x)) = \lambda^n f(x)$. Since the left hand side is bounded this implies $f(x) = 0$ for every x. □

Proof continues

Surjectivity:

Clearly $|x| \leqslant 1 + \varphi_m(x)$ $\forall x \in \mathbb{R}$ and $\forall m \in \mathbb{N}$.

It can be proved that

$$\forall r > 1 \, \forall n \in \mathbb{N} \, \exists C > 0, \, p \in \mathbb{N}$$

such that

$$|\varphi_m^{(n)}(x)| \leqslant C r^m (1 + \varphi_m(x))^p, \ \forall m \, \forall x.$$

From here, we get the convergence in $\mathcal{S}(\mathbb{R})$ of

$$f = -\sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} g \circ \varphi_k$$

for every g and $C_\varphi f - \lambda f = g$.