# Wigner transform and quasicrystals

Joint work with P. Boggiatto, C. Fernández, A. Oliaro

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# Plan of the talk

- Fourier quasicrystals
- Wigner transform and quasicrystals
- The matrix Wigner transform

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- From the mathematical point of view: Yves Meyer (1970s).

## Fourier quasicrystals

By a Fourier quasicrystal we mean a tempered distribution $\mu \in \mathcal{S}'(\mathbb{R}^d)$ of the form $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$ for which $\widehat{\mu} = \sum_{s \in S} b_s \delta_s$, where $\delta_\xi$ is the mass point at $\xi$, $\Lambda$ and $S$ are discrete subsets of $\mathbb{R}^d$.

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## Poisson summation formula

Given a lattice $\Lambda = T(\mathbb{Z}^d)$, where $T$ is an invertible linear map. Then

$$\mu = \sum_{\lambda \in \Lambda} \delta_\lambda \Longrightarrow \widehat{\mu} = \frac{1}{\det T} \sum_{s \in \Lambda^*} \delta_s,$$

where

$$\Lambda^* := (T^*)^{-1}(\mathbb{Z}^d) = \{x \in \mathbb{R}^d : \ \langle x, \lambda \rangle \in \mathbb{Z} \ \ \forall \lambda \in \Lambda\}.$$

Dirac combs have a well-defined periodic structure.
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### Definition

A set $A \subset \mathbb{R}^d$ is said to be *uniformly discrete* (u.d.) if there is $\delta > 0$ such that $|r - s| \geq \delta$ whenever $s, r \in A, s \neq r$.

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- If $\Lambda$ is uniformly discrete, a necessary and sufficient condition for the measure $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$ to be a tempered distribution is that there exists some constant $N \in \mathbb{N}$ such that

$$|a_\lambda| = O(|\lambda|^N)$$

as $|\lambda|$ goes to infinity.

**Theorem (N. Lev, A. Olevskii, 2015))**

*If the support and the spectrum of a measure $\mu$ on $\mathbb{R}$ are uniformly discrete then $\mu$ is a finite sum of Dirac combs, translated and modulated:*

$$\mu = \sum_{j=1}^{N} P_j(t) \sum_{\lambda \in \Lambda} \delta_{\lambda + \theta_j}.$$

$\Lambda$ is a lattice and $P_j(t)$ are trigonometric polynomials.

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- The same result is true in $\mathbb{R}^d$, under the extra assumption that $\mu$ is a positive measure.
- Previous results in this direction: Meyer (1970), A. Córdoba (1989), Kolountzakis-Lagarias (1996).

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A complex measure on $\mathbb{R}^2$ whose support and spectrum are uniformly discrete sets but whose support is not contained in a finite union of translations of a single lattice.

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**Theorem (V.P. Palamodov, 2017)**

*Let $0 \neq \mu \in \mathcal{S}'(\mathbb{R}^d)$ be a tempered distribution on $\mathbb{R}^d$ with support $\Lambda$ and spectrum $\Sigma$. We assume that the sets $\Lambda$ - $\Lambda$ and $\Sigma$ - $\Sigma$ are discrete sets and one of them is uniformly discrete. Then*

*$\Lambda$ is a finite union of translates of a single lattice $L$ and $\Sigma$ is a finite union of translates of the dual lattice $L^*$.*

- Lev, Olevskii (2016): there exists a Fourier quasicrystal whose support and spectrum are discrete closed sets on the real line but with the property that the support contains only finitely many elements of any arithmetic progression. It follows that the support of $\mu$ can not contain any lattice.

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- Explicit examples of quasicrystals that do not have a structure based on the Poisson summation formula: P. Kurasov and P. Sarnak (2020).

**Aim**

Detect Fourier quasicrystals from the information contained in a joint time-frequency representation.

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## Definition

Let $f$, $g \in L^2(\mathbb{R}^d)$ be given. The cross Wigner transform is

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i\omega t}dt, \quad x,\omega \in \mathbb{R}^d.$$

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- $W(f) := W(f,f)$.

## Properties of the Wigner transform

- Covariant property.
  $W(T_u M_\eta f)(x,\omega) = Wf(x-u,\omega-\eta)$, where

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- Moyal's formula.
  For $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ we have

$$\langle W(f_1, g_1), W(f_2, g_2)\rangle = \langle f_1, f_2\rangle \cdot \overline{\langle g_1, g_2\rangle}.$$

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  For $f, \widehat{f} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we have

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- $W(\widehat{f})(x,\omega) = W(f)(-\omega, x)$.

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$$\langle W(\mu, \nu), \phi\rangle = \langle \mu \otimes \overline{\nu}, \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\phi\rangle$$

for any $\phi \in \mathcal{S}(\mathbb{R}^{2d})$, where $\mathcal{F}_2$ denotes the partial Fourier transform with respect to the second variable and $\mathcal{T}_s$ is the symmetric coordinate change defined by

$$\mathcal{T}_s F(x, t) = F(x + \frac{t}{2}, x - \frac{t}{2}), \ \ x, t \in \mathbb{R}^d. \tag{1}$$

### Example (Boggiatto, Fernández, G., Oliaro (2022))

There is a distribution $\mu \in \mathcal{S}'(\mathbb{R})$ whose Wigner transform is supported on a uniformly discrete subset of $\mathbb{R}^2$ even though the support of $\mu$ coincides with $\mathbb{R}$.

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**Ingredients:**

- For every $A \in \mathrm{Sp}(2, \mathbb{R})$ there is a unitary operator $T_A$ acting on $L^2(\mathbb{R})$ such that

$$W\big(T_A f, T_A g\big) = W(f, g) \circ A^* \quad \forall f, g \in L^2(\mathbb{R}).$$

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- The previous relation can be extended to arbitrary tempered distributions.
- Take $\mu = \sum_{n\in\mathbb{Z}} \delta_n$ and $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\theta \in (-\pi, \pi)$. Then $T_A\mu$ is a fractional Fourier transform of $\mu$, and $W(T_A\, \mu)$ is a rotation of $W(\mu)$. The conclusion follows after choosing $\theta$ appropriately.

# Main result

**Theorem (Boggiatto, Fernández, G., Oliaro (2022))**

*Let $\mu \in \mathcal{S}'(\mathbb{R}^d)$ satisfy $W(\mu) = \sum_{(r,s)\in A\times B} c_{r,s}\delta_{(r,s)}$ where $A, B$ are uniformly discrete sets in $\mathbb{R}^d$. Then $\mu$ and $\widehat{\mu}$ are measures. Moreover, the support of $\mu$ is a finite union of translates of a single lattice $L$, while the support of $\widehat{\mu}$ is a finite union of translates of the dual lattice $L^*$.*

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**Lemma**

*Under the hypotheses of the theorem it is fulfilled that supp $\mu \subset A$. Moreover, $\frac{r_1+r_2}{2} \in A$ for any $r_1, r_2 \in$ supp $\mu$. A similar statement holds for $\widehat{\mu}$ and $B$.*

## Remark

*The inclusions obtained above go into the opposite direction with respect to the classical inclusions*

$$\Pi_1(\mathit{supp}\ W\mu) \subset H(\mathit{supp}\ \mu),\ \Pi_2(\mathit{supp}\ W\mu) \subset H(\mathit{supp}\ \widehat{\mu}),$$

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*An immediate consequence of the previous lemma is that the set* $\frac{\text{supp } \mu + \text{supp } \mu}{2}$ *is u.d., as it is a subset of* $A$*. This fact will be crucial in the proof of our theorem. Note that this is not true for arbitrary u.d. sets. For instance* $A = \left\{ n + \frac{1}{|n|} : \ n \in \mathbb{Z} \setminus \{0\} \right\}$ *is u.d. but* $0$ *is an accumulation point of* $\frac{A+A}{2}$.

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From the lemma: $\Lambda :=$ supp $\mu$ and $\Sigma :=$ supp $\widehat{\mu}$ have the property that $\Lambda$ - $\Lambda$ and $\Sigma$ - $\Sigma$ are uniformly discrete.

The statement on the supports now follows from Palamodov's theorem.

# Sketch of the proof that $\mu$ is a measure in the one-dimensional case

$$\mu = \sum_{r \in \mathrm{supp} \mu} \sum_{j=0}^{N} a_r^j \delta_r^{(j)},$$

with $a_r^j \in \mathbb{C}$.

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For any real-valued functions $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R})$ we have, for $\phi = \phi_1 \otimes \phi_2$,

$$\langle W(\mu), \phi_1 \otimes \phi_2 \rangle = \langle \mu_u, \langle \mu_v, \phi_1 \left( \frac{u+v}{2} \right) \overline{\widehat{\phi_2}\,(v-u)} \rangle \rangle$$

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$$= \sum_{j,k=0}^{N} \sum_{\ell=0}^{j} \sum_{m=0}^{k} (-1)^{j+k} \lambda_{j,k}^{\ell,m} \sum_{r,\, s \in \mathrm{supp}\mu} a_s^k \overline{a}_r^j \phi_1^{(\ell+m)} \left( \frac{r+s}{2} \right) \overline{\widehat{\phi_2}}^{(j+k-\ell-m)}(r-s).$$

Fix $r_0 \in \mathrm{supp}\mu$ and choose $\phi_1 \in \mathcal{S}(\mathbb{R})$ compactly supported on a small neighbourhood of $r_0$ and such that $\phi_1^{(n)}(r_0) = 0$ for $n = 0, \ldots, 2N-1$ whereas $\phi_1^{(2N)}(r_0) \neq 0$.

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Then, for any compactly supported smooth function $\phi_2 \in \mathcal{S}(\mathbb{R})$ , we have

$$\langle W(\mu), \phi_1 \otimes \phi_2 \rangle = \frac{1}{2^{2N}} \phi_1^{(2N)}(r_0) \sum_{r,\, s \in D(r_0)} \overline{a_r^N} a_s^N \widehat{\phi_2}(r - s).$$

Here $D(r_0) := \{(r, s) : \ r, s \in \mathrm{supp}\mu; \ \frac{\mathrm{r+s}}{2} = \mathrm{r}_0\}$.

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Here $D(r_0) := \{(r,s) : \ r, s \in \mathrm{supp}\mu; \ \frac{\mathrm{r+s}}{2} = \mathrm{r}_0\}$.

Since $W(\mu)$ is a Radon measure, the right hand side of the previous expression can be estimated by

$$C||\phi_1||_\infty ||\phi_2||_\infty,$$

where the constant $C$ only depends on the (compact) support of $\phi_1 \otimes \phi_2$.

Fix $\psi$ be supported on a small ball with centered at the origin such that $\psi^{(n)}(0) = 0$ for $n = 0, \ldots, 2N-1$ and $\psi^{(2N)}(0) = 1$, and for each $t \geq 1$, we apply the previous inequality to $\phi_1(x) = \psi\left(t(x - x_0)\right)$, whose support shrink as $t$ increases.

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$$t^{2N} \left| \sum_{(r,s)\in D(r_0)} \overline{a_r^N} a_s^N \widehat{\overline{\phi_2}}(r-s) \right| \leq C ||\psi||_\infty ||\phi_2||_\infty,$$

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where $C$ depends on the support of $\psi \otimes \phi_2$. Taking limits as $t$ goes to infinity we conclude that

$$\sum_{(r,s)\in D(r_0)} \overline{a_r^N} a_s^N \widehat{\overline{\phi_2}}(r-s) = 0$$

for every $\phi_2 \in \mathcal{D}(\mathbb{R})$.

The map

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for every $\phi_2 \in \mathcal{S}(\mathbb{R})$. After choosing $\phi_2$ with the property that its Fourier transform is supported on a small compact neighborhood of the origin we get $a_{r_0}^N = 0$.

# Matrix Wigner transform

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**Definition (Bayer, Cordero, Gröchenig, Trapasso (2020))**

Let $T : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a linear isomorphism. Then the *Matrix-Wigner transform* of $\mu, \nu \in \mathcal{S}'(\mathbb{R}^{2d})$ is defined as:

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**Example**

$T = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \Rightarrow W_T(\mu,\nu)(x,\omega) = \mu(x)\overline{\widehat{\nu}(\omega)}$ and we recover the Rihaczek transform.

## Example

- $T = \begin{pmatrix} \frac{1}{2}I & I \\ -\frac{1}{2}I & I \end{pmatrix} \Rightarrow W_T(\mu,\nu) = A(\mu,\nu)$ is the Ambiguity function given by

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- $T = \begin{pmatrix} 0 & I \\ -I & I \end{pmatrix} \Rightarrow W_T(f,g) = V_g f(x,\omega)$ is the Short time Fourier transform defined by

$$V_g f(x,\omega) = W_T(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi it\omega}\,dt.$$

# Information on supports

If $\Psi \in \mathcal{S}'(\mathbb{R}^{2d})$ satisfies that $\Pi_1 \text{supp } \Psi$ or $\Pi_1 \text{supp } \mathcal{F}_2 \Psi$ is a uniformly discrete set then

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$$T^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

**Proposition (Boggiatto, Fernández, G., Oliaro (2022))**

$$\Pi_1 \mathit{supp}\ W_T(\mu,\nu) \text{ u.d.} \Rightarrow \begin{cases} \det A \neq 0 \Longrightarrow \mathit{supp}\ \mu \text{ is u.d,} \\ \det B \neq 0 \Longrightarrow \mathit{supp}\ \nu \text{ is u.d.} \end{cases}$$

$$\Pi_2 \mathit{supp}\ W_T(\mu,\nu) \text{ u.d.} \Rightarrow \begin{cases} \det A \neq 0 \Longrightarrow \mathit{supp}\ \widehat{\nu} \text{ is u.d,} \\ \det B \neq 0 \Longrightarrow \mathit{supp}\ \widehat{\mu} \text{ is u.d.} \end{cases}$$

# The one-variable case

We denote

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and assume that $ab \neq 0$. This condition is satisfied for instance for the matrix defining the ambiguity function or the short time Fourier transform.

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**Theorem**

*Let $\mu, \nu \in \mathcal{S}'(\mathbb{R}) \setminus \{0\}$ satisfy $W_T(\mu, \nu) = \sum_{(r,s)\in A\times B} c_{r,s}\delta_{(r,s)}$ where $A, B$ are u.d. sets. Then $\mu, \widehat{\mu}, \nu, \widehat{\nu}$ are measures supported in u.d. sets.*

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*Let $\mu, \nu \in \mathcal{S}'(\mathbb{R}) \setminus \{0\}$ satisfy $W_T(\mu, \nu) = \sum_{(r,s) \in A \times B} c_{r,s} \delta_{(r,s)}$ where $A, B$ are u.d. sets. Then $\mu, \widehat{\mu}, \nu, \widehat{\nu}$ are measures supported in u.d. sets.*

**Corollary**

*Let $\mu, \nu \in \mathcal{S}'(\mathbb{R}) \setminus \{0\}$ satisfy $W_T(\mu, \nu) = \sum_{(r,s) \in A \times B} c_{r,s} \delta_{(r,s)}$ where $A, B$ are u.d. sets. Then, there are $a, b > 0$ such that $\mu$ is a finite linear combination of time-frequency shifts of $\sum_{n \in \mathbb{Z}} \delta_{na}$ and $\nu$ is a finite linear combination of time-frequency shifts of $\sum_{n \in \mathbb{Z}} \delta_{nb}$.*

# References


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