Ergodic multiplication operators on the Fourier algebra

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Joint work with Enrique Jordá (UPV, Valencia) and Alberto Rodríguez (UJI, Castellón)

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Definition

Let $T\colon E \to E$ a bounded operator, E a Banach space. Put $T_{[n]} = \frac{1}{n}\sum_{k=0}^{n-1} T^k$, we say that T is mean ergodic (ME) if the sequence $(T_{[n]})_n$ is convergent for the SOT topology. If $(T_{[n]})_n$ converges in the operator norm, we say that T is uniformly mean ergodic (UME).

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Let G be a locally compact group and $\mu \in M(G)$. We define the *convolution operator* $\lambda_p(\mu)\colon L_p(G, \mathrm{m}_G) \to L_p(G, \mathrm{m}_G)$

$$\lambda_p(\mu)f(s) = (\mu * f)(s) := \int f(t^{-1}s)d\mu(t), \qquad f \in L_p(G, \mathrm{m}_G),\ s \in G$$

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Consider the augmentation ideal $L_1^0(G) = \left\{ f \in L_1(G)\colon \int f(x)\, dmg(x) = 0 \right\}$ and the operator $\lambda_1^0(\mu) = \lambda_1\big|_{L_1^0(G)}$.

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Note that the equation $\mu * f = f$ has no nontrivial nonconstant solutions in $L^\infty(G)$ precisely when μ is ergodic.

A new perspective: Fourier-Stieltjes transforms

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Definition

The *Fourier-Stieltjes transform* on $\widehat{G}$ is the map $\mathcal{FS}\colon M(\widehat{G}) \to \mathcal{BUC}(G)$, given by

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- $G \cong \sigma(L_1(\widehat{G})) \subset \sigma(M(\widehat{G}))$. If $\mathrm{Gelf}\colon \mathbf{M}(\widehat{\mathbf{G}}) \to \mathbf{C}\left(\sigma\left(\mathbf{M}(\widehat{\mathbf{G}})\right)\right)$ is the Gelfand transform, then for each $\mu \in M(\widehat{G})$,

$$\mathrm{Gelf}(\mu)\big|_G = \mathcal{FS}(\mu).$$

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If G is a locally compact *Abelian* group, the **Fourier algebra**, $\mathbf{A}(\mathbf{G})$, is defined to be the range of $\mathcal{FS}|_{\mathbf{L_1}(\widehat{\mathbf{G}})}$:

$$A(G) = \left\{ \widehat{f} \colon G \to \mathbb{C} : \; f \in L_1(\widehat{G}) \right\}.$$

Since $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$, pointwise multiplication and the norm $\|\widehat{\mathbf{f}}\|_{\mathbf{A(G)}} = \|\mathbf{f}\|_{\mathbf{1}}$, turn $A(G)$ into a **Banach algebra** naturally isomorphic to $L_1(\widehat{G})$.

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If G is a locally compact *Abelian* group the **Fourier-Stieltjes algebra**, $\mathbf{B(G)}$ is the range of $\mathcal{FS}$:

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Since $\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$, pointwise multiplication and the norm $\|\widehat{\mu}\|_{B(G)} = \|\mu\|_{\mathrm{M(G)}}$, turn $B(G)$ into a **Banach algebra** naturally isomorphic to $M(\widehat{G})$.

The **left-regular representation** of a locally compact group G realizes $L_1(G)$ as an algebra of (convolution) operators (and G as a group of *unitary* ones):

$$\lambda_2 \colon \mathbf{L_1}(\widehat{\mathbf{G}}) \to \mathcal{B}(\mathbf{L_2}(\widehat{\mathbf{G}})) \qquad\qquad \lambda_2(f)h = f * h.$$

. . . turned by the Fourier transform into an algebra of multiplication operators:

$$\begin{tikzcd}
L_1(G) \arrow[r, "\lambda_2"] \arrow[d, "\mathcal{F}"'] & \mathcal{B}(L_2(G)) \arrow[d, "\mathcal{F}"] \\
A(\widehat{G}) \subseteq C_0(\widehat{G}) \arrow[r, "\mathrm{Mult}"'] & \mathcal{B}(L_2(\widehat{G}))
\end{tikzcd}$$

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f \arrow[r] \arrow[d] & (h \mapsto f * h) \arrow[d] \\
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$\lambda_2(L_1(G))$ can be seen as an algebra of multiplication operators on $L_2(\widehat{G})$.

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And we can identify $A(G)$ and $B(G)$ as preduals of algebras of operators.

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This approach doesn't require G to be commutative! But doesn't display $A(G)$ and $B(G)$ as algebras of functions on G.

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Second intermezzo: the universal representation

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- Since $\|\pi(f)\| \leq \|f\|_*$, for each $\pi \in \mathrm{Rep}(G)$, there is a one-to-one correspondence between representations of $L_1(G)$ and representations of $C^*(G)$.
- For amenable groups (and this includes Abelian and compact ones) $\|f\|_* = \|\lambda_2(f)\|$. In that case $C^*(G) = \overline{\lambda_2(L_1(G))}^{\|\cdot\|_{\mathrm{op}}}$.

Definition

If $\pi \in \mathrm{Rep}(G)$, the matrix coefficient of π determined by ξ and η is the function $\phi_{\pi,\xi,\eta}\colon G \to \mathbb{C}$

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Theorem (GNS theorem)

If ϕ is a positive functional on a C^*-algebra $\mathcal{C}$ (a positive definite function on a group G), there is representation $\pi\colon \mathcal{C} \to \mathcal{B}(\mathbb{H}_\pi)$ on a Hilbert space $\mathbb{H}_\pi$ (a unitary representation $\pi\colon G \to \mathcal{U}(\mathbb{H}_\pi)$) and $\xi,\ \eta \in \mathbb{H}_\pi$ such that

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\text{Representations of } G \arrow[Rightarrow, r] & \text{Diagonal coefficients} \overset{\text{GNS}}{=} & \text{Positive definite functions on } G
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- $A(G)$ is the closed subalgebra of $B(G)$ generated by coefficients of the regular representation λ_1. Actually, $A(G) = \{f * \tilde{g}\colon f, g \in L_2(G)\}$.

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- **Enveloping von Neumann algebra:** $W^*(G) = \overline{\pi_{\mathrm{univ}}(L_1(G))}^{\mathrm{SOT}}$, i.e. elements of $B(G)^*$ can be seen as operators on $\mathbb{H}_{\mathrm{univ}}$: if $T = (T_\pi)_{\pi \in \mathrm{Rep}(G)} \in W^*(G)$ and $\phi := \phi_{\sigma,\xi,\eta} \in B(G)$,

$$\langle T, \phi \rangle = \langle T_\sigma \xi, \eta \rangle .$$

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Back to ergodicity

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Question: under which conditions are ϕ and the operators $M(\phi))$ and $M^0(\phi)$ mean or uniformly mean ergodic?

$P^1(G) := \{\phi \in B(G) \colon \phi$ positive definite, $\phi(e) = 1\}$.

Definition

Let G be a locally compact group, $\mu \in M(G)$ and $\phi \in B(G)$. We define

$H_\mu := \overline{\langle \mathrm{supp}(\mu) \rangle}$ smallest closed subgroup of G containing the support of μ.

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Key fact: $\|\lambda_1^0(\mu)\| \geq \frac{1}{2}\sup_{x\in G}\|\mu - \mu * \delta_x\|$. It follows that if G is not compact, then $\|\lambda_1^0(\mu)\| = 1$ for every probability measure μ.

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μ *is uniformly ergodic* **if and only if** *there is n_0 s.t. μ^{n_0} is not singular (μ is spread-out).*

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- If μ is ergodic and $S_\mu = \mathrm{supp}(\mu)$, we know that μ ergodic implies that, for some n, $G = \bigcup\limits_{1\leq j,k,\leq n} S_\mu^{-j} S_\mu^{k}$, what is the right analog for $B(G)$?