Ergodic multiplication operators on the Fourier algebra

Jorge Galindo



Joint work with Enrique Jordá (UPV, Valencia) and Alberto Rodríguez (UJI, Castellón)

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Definition

Let $T: E \to E$ a bounded operator, E a Banach space. Put $T_{[n]} = \frac{1}{n} \sum_{k=0}^{n-1} T^k$, we say that T is mean ergodic (ME) if the sequence $(T_{[n]})_n$ is convergent for the SOT topology. If $(T_{[n]})_n$ converges in the operator norm, we say that T is uniformly mean ergodic (UME).



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Let G be a locally compact group and $\mu \in M(G)$. We define the *convolution operator* $\lambda_p(\mu) \colon L_p(G, \mathbf{m}_G) \to L_p(G, \mathbf{m}_G)$

$$\lambda_p(\mu)f(s) = (\mu * f)(s) := \int f(t^{-1}s)d\mu(t), \quad f \in L_p(G, \mathbf{m}_G), \ s \in G$$



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Consider the augmentation ideal $L_1^0(G) = \left\{ f \in L_1(G): \int f(x) dmg(x) = 0 \right\}$ and the operator $\lambda_1^0(\mu) = \lambda_1 |_{L_1^0(G)}$.

Recap: ergodic operators and convolution operators



Let G be a locally compact group and $\mu \in M(G)$. We define the *convolution operator* $\lambda_p(\mu) \colon L_p(G, \mathbf{m}_G) \to L_p(G, \mathbf{m}_G)$

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Note that the equation $\mu * f = f$ has no nontrivial nonconstant solutions in $L^{\infty}(G)$ precisely when μ is ergodic.

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▶ The character group: $\widehat{G} = \{\chi : G \to \mathbb{T} : \chi \text{ a cont. homomorphism}\}$. \widehat{G} is again a locally compact Abelian group.



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 $\widehat{G} = \{\chi_t : \mathbb{Z} \to \mathbb{C}, \chi_t(n) = t^n : t \in \mathbb{T}\}$, $\widehat{\mathbb{Z}} \cong \mathbb{T}$. In general, $G \cong \widehat{\widehat{G}}$, via evaluations.

Definition

The Fourier-Stieltjes transform on \widehat{G} is the map $\mathcal{FS} \colon M(\widehat{G}) \to \mathcal{BUC}(G)$, given by

$$\mathcal{FS}(\mu)(s) = \int_{\widehat{G}} \chi(s) \, d\mu(\chi) =: \widehat{\mu}(s), \quad s \in G.$$

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$$\mathcal{FS}(f) = \int_{\widehat{\mathcal{G}}} \chi(s) f(\chi) \operatorname{dm}_{\widehat{\mathcal{G}}}(\chi) =: \widehat{f}(s), \quad s \in \mathcal{G}.$$

▶ If \mathfrak{A} is a Banach algebra, $\sigma(\mathfrak{A}) \subset \mathfrak{A}^*$ denotes its spectrum, the set of all multiplicative functionals on \mathfrak{A} .

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- ▶ If \mathfrak{A} is a Banach algebra, $\sigma(\mathfrak{A}) \subset \mathfrak{A}^*$ denotes its spectrum, the set of all multiplicative functionals on \mathfrak{A} .
- $G \cong \sigma(L_1(\widehat{G})) \subset \sigma(M(\widehat{G}))$. If Gelf: $M(\widehat{G}) \to C(\sigma(M(\widehat{G})))$ is the Gelfand transform,

then for each $\mu\in M(\widehat{G})$,

$$\operatorname{Gelf}(\mu)|_{G} = \mathcal{FS}(\mu).$$

Definition

If G is a locally compact Abelian group, the Fourier algebra, $\mathbf{A}(\mathbf{G})$, is defined to be the range of $\mathcal{FS}|_{\mathbf{L}_1(\widehat{\mathbf{G}})}$:

$$A(G) = \left\{\widehat{f}: G \to \mathbb{C} : f \in L_1(\widehat{G})\right\}.$$

Since $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$, pointwise multiplication and the norm $\|\widehat{f}\|_{A(G)} = \|f\|_1$, turn A(G) into a Banach algebra naturally isomorphic to $L_1(\widehat{G})$.

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If G is a locally compact Abelian group the Fourier-Stieltjes algebra, B(G) is the range of \mathcal{FS} :

$$B(G) = \left\{ \widehat{\mu} \colon G \to \mathbb{C} : \mu \in M(\widehat{G}) \right\}.$$

Since $\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$, pointwise multiplication and the norm $\|\widehat{\mu}\|_{\mathcal{B}(G)} = \|\mu\|_{\mathsf{M}(G)}$, turn $\mathcal{B}(G)$ into a **Banach algebra** naturally isomorphic to $\mathcal{M}(\widehat{G})$.

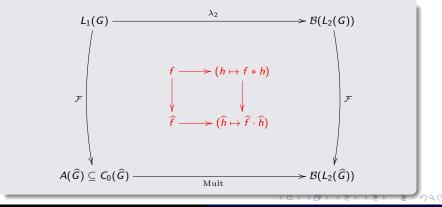
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Intermezzo: the Fourier algebra and algebras of operators on G

The **left-regular representation** of a locally compact group G realizes $L_1(G)$ as an algebra of (convolution) operators (and G as a group of *unitary* ones):

 $\lambda_2 \colon \mathsf{L}_1(\widehat{\mathsf{G}}) \to \mathcal{B}(\mathsf{L}_2(\widehat{\mathsf{G}})) \qquad \qquad \lambda_2(f)h = f * h.$

... turned by the Fourier transform into an algebra of multiplication operators:



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Ergodic multiplication operators on the Fourier algebra

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Jorge Galindo Ergodic multiplication operators on the Fourier algebra

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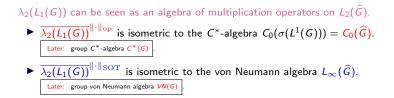
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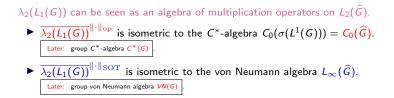
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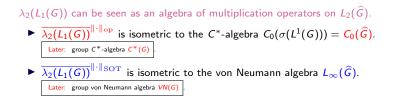


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 $\lambda_2(L_1(G)) \text{ can be seen as an algebra of multiplication operators on } L_2(\widehat{G}).$ $\overline{\lambda_2(L_1(G))}^{\|\cdot\|_{\operatorname{op}}} \text{ is isometric to the } C^*\text{-algebra } C_0(\sigma(L^1(G))) = C_0(\widehat{G}).$ $\overline{L_{\operatorname{ater: group } C^*\text{-algebra } C^*(G)}.$ $\overline{\lambda_2(L_1(G))}^{\|\cdot\|_{\operatorname{SOT}}} \text{ is isometric to the von Neumann algebra } L_\infty(\widehat{G}).$

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- A(G) is isometric to the predual of $\overline{\lambda_2(L_1(G))}^{\|\cdot\|_{\text{SOT}}}$.
- B(G) is isometric to the dual of $\overline{\lambda_2(L_1(G))}^{\|\cdot\|_{\text{op}}}$.

Later: group von Neumann algebra VN(G)

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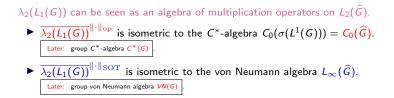
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This approach doesn't require G to be commutative! But doesn't display A(G) and B(G) as algebras of functions on G.

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Let $\operatorname{Rep}(G) = \{\pi \colon G \to \mathcal{U}(\mathbb{H}_{\pi}) \colon \mathbb{H}_{\pi} \text{ a Hilbert space}, \pi \text{ a cyclic unitary representation }\}$, and $\mathbb{H}_{\operatorname{univ}} := \bigoplus_{\pi \in \operatorname{Rep}(G)} \mathbb{H}_{\pi}$, we define:

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Recall that, for $\pi \in \text{Rep}(G)$, always induces, averaging, a representation on $L^1(G)$.

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• $\pi_{\text{univ}}: \mathcal{G} \to \mathcal{U}(\mathbb{H}_{\text{univ}})$, the universal representation of $\mathcal{G}, \pi = \bigoplus_{\pi \in \text{Rep}(G)} \pi$.

The group C^* -algebra, $C^*(G)$ is the completion of $L_1(G)$ under the seminorm $||f||_* = ||\pi_{univ}(f)||$.

Recall that, for $\pi \in \text{Rep}(G)$, always induces, averaging, a representation on $L^1(G)$.

- Since ||π(f)|| ≤ ||f||_{*}, for each π ∈ Rep(G), there is a one-to-one correspondence between representations of L₁(G) and representations of C^{*}(G).
- For amenable groups (and this includes Abelian and compact ones) $||f||_* = ||\lambda_2(f)||$. In that case $C^*(G) = \overline{\lambda_2(L_1(G))}^{\|\cdot\|_{\text{op}}}$.



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If $\pi \in \operatorname{Rep}(G)$, the matrix coefficient of π determined by ξ and η is the function $\phi_{\pi,\xi,\eta} \colon G \to \mathbb{C}$

 $\phi_{\pi,\xi,\eta}(g) = \langle \pi(g)\xi,\eta \rangle.$

When $\xi = \eta$ the coefficient is said to be diagonal.



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Theorem (GNS theorem)

If ϕ is a positive functional on a C^{*}-algebra C (a positive definite function on a group G), there is representation $\pi \colon C \to \mathcal{B}(\mathbb{H}_{\pi})$ on a Hilbert space \mathbb{H}_{π} (a unitary representation $\pi \colon G \to \mathcal{U}(\mathbb{H}_{\pi})$) and $\xi, \eta \in \mathbb{H}_{\pi}$ such that

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Definition

If $\pi \in \operatorname{Rep}(G)$, the matrix coefficient of π determined by ξ and η is the function $\phi_{\pi,\xi,\eta} \colon G \to \mathbb{C}$

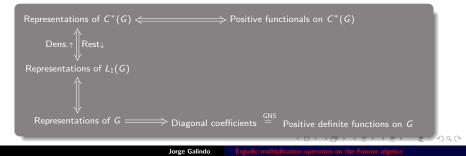
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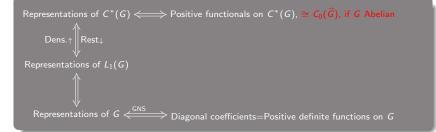
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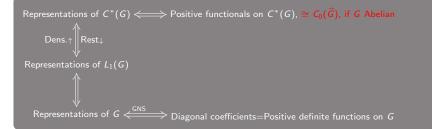
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Definition (Eymard, 1964)



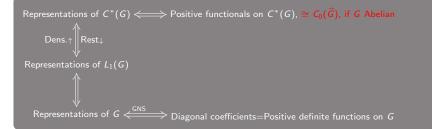
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Definition (Eymard, 1964)

We define B(G) as the multiplication algebra of all coefficients of unitary representations of G. B(G) equipped with the norm that makes B(G) = (C*(G))*.



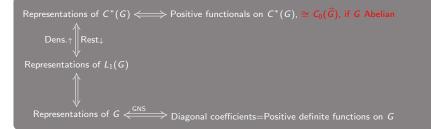
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Definition (Eymard, 1964)

- We define B(G) as the multiplication algebra of all coefficients of unitary representations of G. B(G) equipped with the norm that makes B(G) = (C*(G))*.
- A(G) is the closed subalgebra of B(G) generated by coefficients of the regular representation λ_1 . Actually, $A(G) = \{f * \tilde{g} : f, g \in L_2(G)\}$.



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- Enveloping von Neumann algebra: $W^*(G) = \overline{\pi_{\text{univ}}(L_1(G))}^{\text{SOT}}$, i.e. elements of $B(G)^*$ can be seen as operators on \mathbb{H}_{univ} : if $T = (T_\pi)_{\pi \in \text{Rep}(G)} \in W^*(G)$ and $\phi := \phi_{\sigma,\xi,\eta} \in B(G)$,

$$\langle T, \phi \rangle = \langle T_{\sigma} \xi, \eta \rangle.$$

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Let G be a locally compact group. We define:

• The augmentation ideal: $A^0(G) = \{u \in A(G) : u(e) = 0\}.$

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Question: under which conditions are ϕ and the operators $M(\phi)$) and $M^0(\phi)$ mean or uniformly mean ergodic?

The operators $\lambda_1(\mu)$ and $M(\phi)$)

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$P^1(G) := \{ \phi \in B(G) \colon \phi \text{ positive definite, } \phi(e) = 1 \}.$

Definition

Let G be a locally compact group, $\mu \in M(G)$ and $\phi \in B(G)$. We define

 $\begin{array}{l} H_{\mu} := \overline{\langle \operatorname{supp}(\mu) \rangle} & \text{smallest closed subgroup of } G \text{ containing the support of } \mu. \\ H_{\phi} := \{ x \in G : \phi(x) = 1 \}. \end{array}$

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Let G be a locally compact group and let $\mu \in M(G)$ be a probability measure. Then

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 $M(\phi)$ is mean ergodic $\iff H_{\phi}$ is open.

G a locally compact group. μ prob. measure, $\phi \in P^1(G)$.

Definition

Jorge Galindo Ergodic multiplication operators on the Fourier algebra

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Let $\mu \in M(G)$ and $\phi \in B(G)$. Then:

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- If μ is completely mixing, then μ is strictly aperiodic.

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Uniform mean ergodicity of probabilities and pos. definite functions

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Theorem

If μ is uniformly ergodic, then G is compact.

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If μ is uniformly ergodic, then G is compact.

Key fact: $\|\lambda_1^0(\mu)\| \ge \frac{1}{2} \sup_{x \in G} \|\mu - \mu * \delta_x\|.$

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Theorem

If μ is uniformly ergodic, then G is compact.

Key fact: $\|\lambda_1^0(\mu)\| \ge \frac{1}{2} \sup_{x \in G} \|\mu - \mu * \delta_x\|$. It follows that if *G* is not compact, then $\|\lambda_1^0(\mu)\| = 1$ for every probability measure μ .

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Let G be amenable. If ϕ is uniformly ergodic, then G is discrete.

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Key fact: $||M^0(\phi)|| \ge \frac{M}{2} \sup_{\alpha} ||\phi - \phi \cdot f_{\alpha}||$, where $(f_{\alpha}) \subseteq A(G)$ is a net such that $\lim_{\alpha} ||uf_{\alpha} - f_{\alpha}|| = 0$. It follows that, if G is amenable and nondiscrete, there is C > 0 such that $||M^0(\phi)|| \ge 1/M$ for every $\phi \in P^1(G)$.

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 μ is uniformly ergodic if and only if there is n_0 s.t. μ^{n_0} is not singular (μ is spread-out).

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G amenable. ϕ is uniformly ergodic if and only if there is n_0 s.t. $d(\phi^{n_0}, A(G)) < 1$ (ϕ is spread-out).

G a locally compact group, μ a probability measure and $\phi \in P^1(G)$.

 $\sigma_{M(G)}(\mu) = \sigma\left(\lambda_1^0(\mu)
ight) \cup \{1\}$

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- Is it true that $\sigma(\phi) = \sigma(M(\phi))$ when G is **not** amenable?
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- Is it true that $\sigma(\phi) = \sigma(M(\phi))$ when G is **not** amenable?
- If amenability is not assumed, is it true that φ is uniformly ergodic if and only if φ is spread-out?
- If μ is ergodic and $S_{\mu} = \operatorname{supp}(\mu)$, we know that μ ergodic implies that, for some $n, G = \bigcup_{1 \le j,k, \le n} S_{\mu}^{-j} S_{\mu}^{k}$, what is the right analog for B(G)?