A comparison of two ways to generalize Denjoy-Carleman ultradifferentiable classes

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(joint work with D. N. Nenning and G .Schindl)

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Tale of two cities: Vienna

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- Ultradifferentiable classes defined by weight matrices:

G. Schindl. Exponential laws for classes of Denjoy-Carleman-differentiable mappings, 2013. PhD Thesis, Universität Wien.
A. Rainer and G. Schindl. Composition in ultradifferentiable classes. Studia Math. 2014, 224, 97–131.
G. Schindl. The convenient setting for ultradifferentiable mappings of Beurling- and Roumieu-type defined by a weight matrix. Bull. Belg. Math. Soc. Simon Stevin 22 (2015), no. 3, 471–510.
A. Rainer and G. Schindl. Equivalence of stability properties for ultradifferentiable function classes. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A. Matemáticas, 110(1):17–32, 2016.
G. Schindl. Characterization of ultradifferentiable test functions defined by weight matrices in terms of their Fourier transform. Note di Matematica, 36(2):1–35, 2016.
A. Rainer and G. Schindl. Extension of Whitney jets of controlled growth. Math. Nachr. 290 (2017), no. 14-15, 2356–2374.
A. Rainer and G. Schindl. On the Borel mapping in the quasianalytic setting. Math. Scand., 121(2):293–310, 2017.
C. Esser and G. Schindl. How far is the Borel map from being surjective in quasianalytic ultradifferentiable classes? J. Math. Anal. Appl. 466 (2018), no. 1, 986–1008.
S. Fürdös, D.N. Nenning, A. Rainer and G. Schindl. Almost analytic extensions of ultradifferentiable functions with applications to microlocal analysis. J. Math. Anal. Appl. 2020, 481.
C. Boiti, D. Jornet, A. Oliaro and G. Schindl. Nuclear global spaces of ultradifferentiable functions in the matrix weighted setting. Banach J. Math. Anal. 15 (2021), no. 1, 1–39.

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Tale of two cities: Novi Sad

- S. Pilipović, N. Teofanov and F. Tomić ultradifferentiable classes:

S. Pilipović, N. Teofanov, and F. Tomić. On a class of ultradifferentiable functions. Novi Sad J. Math. 2015, 45, 125–142.
S. Pilipović, N. Teofanov, and F. Tomić. Beyond Gevrey regularity. J. Pseudo-Differ. Oper. Appl. 2016, 7, 113–140.
F. Tomić. A New Type of Regularity with Applications to the Wave Front Sets = Nova vrsta regularnosti sa primenama na talasni front. Thesis (Ph.D.)–University of Novi Sad (Serbia). 2016.
N. Teofanov, and F. Tomić. Ultradifferentiable functions of class $M^{\tau,\sigma}$ and microlocal regularity, Generalized functions and Fourier analysis. In Advances in Partial Differential Equations; Birkhäuser, Basel, Switzerland, 2017; pp. 193–213.
N. Teofanov, and F. Tomić. Inverse closedness and localization in extended Gevrey regularity. J. Pseudo-Differ. Oper. Appl. 8 (2017), no. 3, 411–421.
S. Pilipović, N. Teofanov, and F. Tomić. Beyond Gevrey regularity: Superposition and propagation of singularities. Filomat 2018 , 32, 2763–2782.
S. Pilipović, N. Teofanov, and F. Tomić. Regularities for a new class of spaces between distributions and ultradistributions. Sarajevo J. Math. 14(27) (2018), no. 2, 251–263.
S. Pilipović, N. Teofanov, and F. Tomić. Paley-Wiener theorem in extended Gevrey regularity. J. Pseudo-Differ. Oper. Appl. 2020 , 11, 593–612.
S. Pilipović, N. Teofanov, and F. Tomić. Boundary values in ultradistribution spaces related to extended Gevrey regularity. Mathematics, 9(1), 2021.

Denjoy-Carleman ultradifferentiable classes

Denjoy-Carleman classes. Let $M = (M_p)_{p\in\mathbb{N}} \in \mathbb{R}_{>0}^{\mathbb{N}}$ be a sequence of positive real numbers, $h > 0$ and $K \subset\subset \mathbb{R}^d$ be a regular compact set. By $\mathcal{E}_{M,h}(K)$ we denote the Banach space of functions $f \in \mathcal{C}^\infty(K)$ such that

$$||f||_{\mathcal{E}_{M,h}(K)} = \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in K} \frac{|\partial^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty.$$

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Let U be an open set of $\mathbb{R}^d$ and $K \subset\subset U$. We define the spaces of Roumieu, and resp. of Beurling, type:

$$\mathcal{E}_{\{M\}}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}_{M,h}(K), \qquad \mathcal{E}_{(M)}(U) = \varprojlim_{K\subset\subset U} \varprojlim_{h\to 0} \mathcal{E}_{M,h}(K).$$

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Note that the case $\sigma = 1$ we have the classical Gevrey classes with index τ.

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- In their results they use an adapted version of H. Komatsu conditons:

$\overline{(\mathrm{M.2})'}$ $\exists\, D > 0$, such that $\forall p \in \mathbb{N}$ $M^{\tau,\sigma}_{p+1} \leq D^{p^\sigma} M^{\tau,\sigma}_{p}$.

$\overline{(\mathrm{M.2})}$ $\exists\, C > 0$, such that $\forall p, q \in \mathbb{N}$ $M^{\tau,\sigma}_{p+q} \leq C^{p^\sigma + q^\sigma} M^{\tau 2^{\sigma-1},\sigma}_{p} M^{\tau 2^{\sigma-1},\sigma}_{q}$.

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- The space $\mathcal{E}_{[1,2]}(U)$ is explicitly used in the study of strictly hyperbolic equations, see [1] for details.

[1] M.Cicognani and D.Lorentz. Strictly hyperbolic equations with coefficient slow-regular in time and smooth in space, J.Pseudo-Differ. Oper. Appl. 9 (3) (2018) 643 – 675.

Ultradifferentiable classes defined by weight matrices

We say that $\mathcal{M} := \{M^{(\lambda)} \in \mathbb{R}^{\mathbb{N}}_{>0} : \lambda \in \mathbb{R}_{>0}\}$ a *weight matrix* if

$$\forall\, \lambda \leq \mu \quad \forall\, p \in \mathbb{N} \quad M_p^{(\lambda)} \leq M_p^{(\mu)}.$$

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Classes defined by a weight matrix. For $U \subseteq \mathbb{R}^d$ non-empty open set, we consider classes defined by weight matrices of Roumieu type $\mathcal{E}_{\{\mathcal{M}\}}$ and of Beurling type $\mathcal{E}_{(\mathcal{M})}$ as follows

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Weight matrices allow us to give a unified treatment of the spaces defined by a single weight sequences, Denjoy-Carleman ultradifferentiable classes, and by a single weight functions, Braun-Meise-Taylor classes. Moreover, proofs can be transferred from one context to the other.

Purpose of the talk

In the introduction of [A] the authors claimed that:

> *"Let us comment on another very interesting concept of construction of a large class of ultradistribution spaces. In [B, C, D] and several other papers the authors consider sequences of the form $k!M_k$, where they presume a fair number of conditions on M_k and discuss in details their relations. For example, consequences of the composition of ultradifferentiable functions determined by different classes of such sequences are discussed. Moreover, they consider weighted matrices, that is, a family of sequences of the form $k!M_k^\lambda$, $k \in \mathbb{N}$, $\lambda \in \Lambda$ (partially ordered and directed set), and make the unions, again considering various properties such as compositions and boundary values [...]* **The main reason why our classes are not covered by the quoted papers is the factor $h^{|\alpha|^\sigma}$, $\sigma > 1$, in the seminorm.**"

[A] S. Pilipović, N. Teofanov, and F. Tomić. Boundary values in ultradistribution spaces related to extended Gevrey regularity. Mathematics, 9(1), 2021.

[B] S. Fürdös, D.N. Nenning, A. Rainer and G. Schindl. Almost analytic extensions of ultradifferentiable functions with applications to microlocal analysis. J. Math. Anal. Appl. 2020, 481

[C] A. Kriegl, P.W. Michor and A. Rainer. The convenient setting for quasianalytic Denjoy–Carleman differentiable mappings. J. Funct. Anal. 2011, 261, 1799–1834.

[D] A. Rainer and G. Schindl. Composition in ultradifferentiable classes. Studia Math. 2014, 224, 97–131.

P.T.T. classes as classes defined by weight matrix

Let the parameters $\tau > 0$ and $\sigma > 1$ be given. We consider the weight matrix:

$$\mathcal{M}^{\tau,\sigma} = \{(c^{p^\sigma} p^{\tau p^\sigma})_{p\in\mathbb{N}} \quad : \quad c > 0\}.$$

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Theorem [J.J.-G. D.N. Nenning, G. Schindl] As locally convex vector spaces we get

$$\mathcal{E}_{\{\tau,\sigma\}}(U) = \mathcal{E}_{\{\mathcal{M}^{\tau,\sigma}\}}(U), \qquad \mathcal{E}_{(\tau,\sigma)}(U) = \mathcal{E}_{(\mathcal{M}^{\tau,\sigma})}(U).$$

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The same hols for other classes (Global bounds, test functions, ultraholomorphic, etc).

Properties of the weight matrix

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Proposition [J.J.-G. D.N. Nenning, G. Schindl]

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Remark about the properties of the weight matrix

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This contradicts the claim also in [A]:

> *"[...] The main reason why our classes are not covered by the quoted papers is the factor $h^{|\alpha|^\sigma}$, $\sigma>1$, in the seminorm.* **For that reason our conditions on the weight sequence ($\overline{\text{(M.2)}'}$ and $\overline{\text{(M.2)}}$ given below) differ from the corresponding ones in the quoted papers.** *[...]"*

[A] S. Pilipović, N. Teofanov, and F. Tomić. Boundary values in ultradistribution spaces related to extended Gevrey regularity. Mathematics, 9(1), 2021.

Consequences

Corollary [J.J.-G. D.N. Nenning, G. Schindl] For all open set $U \subseteq \mathbb{R}^d$:

(a) By (i) the classes $\mathcal{E}_{\{\mathcal{M}^{\tau,\sigma}\}}(U)$ and $\mathcal{E}_{(\mathcal{M}^{\tau,\sigma})}(U)$ contain the real analytic functions in U and the restriction to U of the entire functions in $\mathbb{C}^d$.

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(g) By (i), (ii) and (iii) we have that $\widetilde{\mathcal{M}^{\tau,\sigma}}$ is a *regular* weight matrix. So all results from about almost analytic extensions and microlocal analysis of [1, Sect. 1-6] can be applied to the classes $\mathcal{E}_{\{\mathcal{M}^{\tau,\sigma}\}}$ and $\mathcal{E}_{(\mathcal{M}^{\tau,\sigma})}$.

[1] S. Fürdös, D.N. Nenning, A. Rainer and G. Schindl. Almost analytic extensions of ultradifferentiable functions with applications to microlocal analysis. J. Math. Anal. Appl. 2020, 481.

Weight matrices are the natural representation

The associated weight functions to the sequences $M^{(c,\tau,\sigma)} = (c^{p^\sigma} p^{\tau p^\sigma})_{p\in\mathbb{N}}$ are pairwise non-equivalent. In particular, if $0 < c_1 < c_2$, we have that

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Corollary [J.J.-G. D.N. Nenning, G. Schindl] Neither $\mathcal{E}_{\{\mathcal{M}^{\tau,\sigma}\}}(U)$ nor $\mathcal{E}_{(\mathcal{M}^{\tau,\sigma})}(U)$ coincides (as vector space) with $\mathcal{E}_{\{M\}}(U)$, $\mathcal{E}_{\{\omega\}}(U)$, or, respectively, $\mathcal{E}_{(M)}(U)$, $\mathcal{E}_{(\omega)}(U)$ for any weight sequence M or any weight function ω.

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Weight matrices are the natural context to study these classes.

Ultradifferentiable classes beyond geometric factors

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Classes beyond geometric factors. Let $M = (M_p)_{p\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{>0}$, $\Phi = (\Phi_p)_{p\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{\geq 0}$, $h > 0$ and $K \subset\subset \mathbb{R}^d$ be a regular compact set. By $\mathcal{E}_{M,\Phi,h}(K)$ we denote the space of functions $f \in \mathcal{C}^\infty(K)$ such that

$$||f||_{\mathcal{E}_{M,\Phi,h}(K)} = \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in K} \frac{|\partial^\alpha f(x)|}{h^{\Phi_{|\alpha|}} M_{|\alpha|}} < \infty.$$

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Let U be an open set of $\mathbb{R}^d$ and $K \subset\subset U$. We define the spaces of Roumieu, and resp. of Beurling, type:

$$\mathcal{E}_{\{M,\Phi\}}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}_{M,\Phi,h}(K), \qquad \mathcal{E}_{(M,\Phi)}(U) = \varprojlim_{K\subset\subset U} \varprojlim_{h\to 0} \mathcal{E}_{M,\Phi,h}(K).$$

Equality result

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Theorem [J.J.-G. D.N. Nenning, G. Schindl] If the sequence Φ satisfies the condition

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Other properties: Depend on the relation between Φ and M.

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Prop. [S. Pilipović, N. Teofanov, and F. Tomić, 2016] Let $\sigma \geq 1$ and $\tau > 0$. We have that :

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A tricky definition due to the quantifiers. In

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there are two parameters h (local) depending on the compact and τ (global) independent from the compact. However, in the definition of $\mathcal{E}_{\{\mathcal{M}^\sigma\}}(U)$ there is only one local parameter.

Comparison results

Theorem [J.J.-G. D.N. Nenning, G. Schindl] Let $\mathcal{M}$ be a weight matrix and suppose

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Theorem [J.J.-G. D.N. Nenning, G. Schindl] Let $U, V \subseteq \mathbb{R}^d$ be open, and $\overline{V} \subset\subset U$. Then as locally convex vector spaces we get

$$\mathcal{E}_{\infty,\sigma}(U) \hookrightarrow \mathcal{E}_{\{\mathcal{M}^\sigma\}}(U) \hookrightarrow \mathcal{E}_{\infty,\sigma}(V), \qquad \mathcal{E}_{0,\sigma}(U) = \mathcal{E}_{(\mathcal{M}^\sigma)}(U).$$

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[1] S. Fürdös, D.N. Nenning, A. Rainer and G. Schindl. Almost analytic extensions of ultradifferentiable functions with applications to microlocal analysis. J. Math. Anal. Appl. 2020, 481.

Last comments

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Even if the classes $\mathcal{E}_{\infty,\sigma}(U)$ and $\mathcal{E}_{\{\mathcal{M}^\sigma\}}(U)$ do not coincide, one can apply the mixed results between $\mathcal{E}_{\{\tau,\sigma\}}(U) = \mathcal{E}_{\{\mathcal{M}^{\tau,\sigma}\}}(U)$ and $\mathcal{E}_{\{\tau',\sigma\}}(U) = \mathcal{E}_{\{\mathcal{M}^{\tau',\sigma}\}}(U)$ from the theory of ultradifferentiable classes defined by weight matrices. In other words, the class $\mathcal{E}_{\infty,\sigma}(U)$ can be studied with the information about weight matrices.

Thank you for your attention!