On supercyclic composition operators

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Enrique Jordá

Joint work with
M.J. Beltrán Menéu and M. Murillo-Arcila

and with A.A. Albanese and C. Mele

IUMPA
Instituto Universitario de Matemática
Pura y Aplicada

UNIVERSITAT
POLITÈCNICA
DE VALÈNCIA

Aim of the study

Let X be a topological Hausdorff space with infinite cardinal. Let $E \hookrightarrow (C(X), \tau_p)$ be a separable ∞-dim l.c.s s.t $\{\delta_x\}_{x \in X} \subseteq E'$ lin.ind. Let $w : X \to \mathbb{C}$ (a **multiplier**) and $\varphi : X \to X$ (a **symbol**) be continuous.

Aim

To study **weak forms of supercyclicity** of the **weighted composition operator** $C_{w,\varphi} : E \to E$,

$$C_{w,\varphi}(f) = w(f \circ \varphi), \ f \in E,$$

when it is well defined and continuous.

Examples of E spaces:

- $H(\mathbb{D}), H(\mathbb{D} \setminus \{0\}), H(\mathbb{C} \setminus \{0\}), A(\mathbb{D}), Lip_\alpha(\mathbb{D}), 0 < \alpha \leq 1.$
- $C(\partial\mathbb{D}), C(\overline{\mathbb{D}}), C^m(\mathbb{R})$

Linear dynamics. Basic definitions

Given an operator T on a t.v.s (F,τ):

$f \in F$ is **periodic** if $\exists n \in \mathbb{N}$ such that $T^n f = f$, $T^n := T \circ \overset{n)}{\cdots} \circ T$.
$f \in F$ is a **fixed point** if $Tf = f$.

Dynamical definitions

- T τ-**hypercyclic:** $\exists f \in F$ (τ-hypercyclic vector) s.t. $\mathrm{Orb}(T,f) := \{T^n f : \ n = 0,1,\dots\} = \{f, Tf, T^2 f, \dots\}$ is dense.
- T τ-**supercyclic:** $\exists f \in F$ (τ-supercyclic vector) s.t. $\mathrm{Orb}(T, \mathrm{span}\{f\}) = \{\lambda T^n f : \lambda \in \mathbb{C}, n = 0,1,\dots\}$ is dense in F.
- T τ-**cyclic:** $\exists f \in F$ (a τ-cyclic vector) s.t. $\mathrm{span}\{\mathrm{Orb}(T,f)\} = \mathrm{span}\{T^n f : \ n = 0,1,\dots\}$ is dense in F.

$\tau =$ weak topology: T is weakly hypercyclic/supercyclic/cyclic.

$\tau = \tau_p$: T is pointwise hypercyclic/supercyclic/cyclic.

$\tau =$ strong topology: T is hypercyclic/supercyclic/cyclic.

Linear dynamics

In a Banach space, we have :

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{hypercyclic} \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow, shift right] & \text{weakly hypercyclic} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[d, Rightarrow, shift right] \\
\text{supercyclic} \arrow[u, Rightarrow, shift right, "/" marking] \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow, shift right] & \text{weakly supercyclic} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[u, Rightarrow, shift right, "/" marking] \arrow[d, Rightarrow, shift right] \\
\text{cyclic} \arrow[u, Rightarrow, shift right, "/" marking] \arrow[r, Leftrightarrow] & \text{weakly cyclic} \arrow[u, Rightarrow, shift right, "/" marking]
\end{tikzcd}$$

Section 1

Dynamics of $C_{w,\varphi}$ on Banach spaces of continuous functions

Background: Dynamics of $C_{w,\varphi}$ on $H(\mathbb{D})$

(Weak) Hypercyclicity and (weak) supercyclicity:

Yousefi and Rezaei (2007), Kamali, Hedayatian and Khani Robati (2010). Bès (2014) improved their results:

Theorem (Bès, 2014)

Let φ be a holomorphic self map of a simply connected plane domain Ω and let $w \in H(\Omega)$. The following are equivalent:

a) The operator $C_{w,\varphi}$ is weakly supercyclic on $H(\Omega)$.

b) w is zero-free and φ is univalent and without fixed points.

c) The operator $C_{w,\varphi}$ is mixing on $H(\Omega)$ ($\Rightarrow C_{w,\varphi}$ hypercyclic)

a) $\Rightarrow$ b) is satisfied even for Ω an arbitrary planar domain.

Background: Dynamics of $C_{w,\varphi}$ on $Y \subseteq H(\mathbb{D})$

Let $Y \subseteq H(\mathbb{D})$ be a Banach space s.t. every $f \in Y$ has a continuous extension to $\overline{\mathbb{D}}$ and δ_z is bounded $\forall z \in \partial\mathbb{D}$.
Ex: $Y = A(\mathbb{D}), Y = Lip_\alpha(\mathbb{D}), 0 < \alpha \leq 1.$

Moradi, Khani Robati, Hedayatian (2017)*

Let $\varphi, w \in Y$, $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, and let $a \in \overline{\mathbb{D}}$ be a fixed point of φ s.t. $w(a) \neq 0$. If $C_{w,\varphi}$ is weakly supercyclic on Y, then

$$\left\{ \frac{\prod_{m=0}^{n} w(\varphi^m(z))}{w^n(a)},\ n \in \mathbb{N} \right\} \text{ is unbounded } \forall z \in \overline{\mathbb{D}} \setminus \{a\}.$$

* They prove the result for a class of weighted composition operators.

Moradi, Khani Robati, Hedayatian (2017)

Given $\varphi \in Y$, C_φ is never weakly supercyclic on Y.

Supercyclicity of $C_{w,\varphi}$ on spaces of continuous functions

Proposition (Beltrán-Menéu, J., Murillo-Arcila)

If $C_{w,\varphi} : E \to E$, $E \hookrightarrow (C(X), \tau_p)$, is τ_p-supercyclic, then:

i) w is zero-free

ii) φ is univalent

iii) $\forall \tau_p$-supercyclic function f and $\forall z_1 \neq z_2 \in X$, such that $\{\delta_{z_1}, \delta_{z_2}\}$ is linearly independent

$$\overline{\left\{ \frac{\prod_{m=0}^{n-1} w(\varphi^m(z_1)) f(\varphi^n(z_1))}{\prod_{m=0}^{n-1} w(\varphi^m(z_2)) f(\varphi^n(z_2))},\ n \in \mathbb{N} : f(\varphi^n(z_2)) \neq 0 \right\}} = \mathbb{C}$$

If in addition, $\varphi^n(z_1) \to a$ and $\varphi^n(z_2) \to b$, $a, b \in X$ fixed points, then:

$$\overline{\left\{ \frac{\prod_{m=0}^{n} w(\varphi^m(z_1))}{\prod_{m=0}^{n} w(\varphi^m(z_2))},\ n \in \mathbb{N} \right\}} = \mathbb{C}.$$

Proof:

Let $z_1, z_2 \in X$, $\{\delta_{z_1}, \delta_{z_2}\}$ is linearly independent. The mapping $F : E \to \mathbb{C}^2$, $F(g) = (g(z_1), g(z_2))$ is τ_p-continuous and surjective. Thus, if $f \in E$ is a τ_p-supercyclic vector, as $C^n_{w,\varphi} f = \left(\prod_{m=0}^{n-1} w \circ \varphi^m\right) f \circ \varphi^n$, we get

$$\left\{\left(\lambda \prod_{m=0}^{n-1} w(\varphi^m(z_1)) f(\varphi^n(z_1)), \lambda \prod_{m=0}^{n-1} w(\varphi^m(z_2)) f(\varphi^n(z_2))\right) : \lambda \in \mathbb{C}, n \in \mathbb{N}\right\}$$

is dense in $\mathbb{C}^2$. Given $c \in \mathbb{C}\setminus\{0\}$, $\exists (n_k)_k$, s.t. $\lambda_{n_k} \neq 0$, $f(\varphi^{n_k}(z_i)) \neq 0$ for $i = 1, 2$, and

$$\left(\lambda_{n_k} \prod_{m=0}^{n_k-1} w(\varphi^m(z_1)) f(\varphi^{n_k}(z_1)), \lambda_{n_k} \prod_{m=0}^{n_k-1} w(\varphi^m(z_2)) f(\varphi^{n_k}(z_2))\right) \to (c, 1).$$

As a consequence,

$$\lim_k \frac{\prod_{m=0}^{n_k-1} w(\varphi^m(z_1)) f(\varphi^{n_k}(z_1))}{\prod_{m=0}^{n_k-1} w(\varphi^m(z_2)) f(\varphi^{n_k}(z_2))} = c$$

Supercyclicity of $C_{w,\varphi}$ on spaces of continuous functions

Theorem (Beltrán-Menéu, J., Murillo-Arcila)

Let X be compact and let $E \hookrightarrow (C(X), \| \|_\infty)$ be Banach and containing a nowhere vanishing function. Then, $C_{w,\varphi} : E \to E$ is never weakly supercyclic.

Corollary

$C_{w,\varphi}$ is not weakly supercyclic on $A(\mathbb{D})$, neither on $Lip_\alpha(\mathbb{D}), 0 < \alpha \leq 1$.

Proof:

- $C_{w,\varphi} : E \to E$ is weakly supercyclic $\Rightarrow$ the set of weakly supercyclic vectors is norm dense (Sanders, 2004), and thus, $\| \, \|_\infty$-dense $\Rightarrow$

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- $\exists\ f$ weakly supercyclic and $\epsilon > 0$ s.t. $|f(z)| \geq \epsilon\ \forall z \in X$ ($\exists$ nowhere vanishing functions).

Proof:

- $C_{w,\varphi}: E \to E$ is weakly supercyclic $\Rightarrow$ the set of weakly supercyclic vectors is norm dense (Sanders, 2004), and thus, $\| \, \|_\infty$-dense $\Rightarrow$
- $\exists\ f$ weakly supercyclic and $\epsilon > 0$ s.t. $|f(z)| \geq \epsilon\ \forall z \in X$ ($\exists$ nowhere vanishing functions).
- $M_w: E \to E$ is not weakly supercyclic. So, we can assume $\exists z_0 \in X$ s.t. $z_1 = \varphi(z_0) \neq z_0$. Therefore, $\exists C > 0$ such that

$$\left| \frac{\prod_{m=0}^{n-1} w(\varphi^m(z_1)) f(\varphi^n(z_1))}{\prod_{m=0}^{n-1} w(\varphi^m(z_0)) f(\varphi^n(z_0))} \right| = \left| \frac{w(\varphi^n(z_0)) f(\varphi^{n+1}(z_0))}{w(z_0) f(\varphi^n(z_0))} \right| \leq C$$

$\forall n \in \mathbb{N}$, a contradiction.

Supercyclicity of C_φ on spaces of one variable real functions

Theorem ♣ (Beltrán-Menéu, J., Murillo-Arcila)

Let $E \hookrightarrow (C(X), \tau_p)$, asume $\{\delta_x : \ x \in X\} \subseteq E'$ to be linearly independent. Any of the following conditions implies $C_{w,\varphi} : E \to E$, is not τ_p-supercyclic:

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i) φ has two fixed points $\{z_1, z_2\}$.

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ii) $\exists \varphi^n(z_1) \to z_0$, $z_0, z_1 \in X$, different.

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iv) X is compact, φ has a fixed point z_1 such that $|w(z)| \leq |w(z_1)|$ $\forall z \in X$.

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v) φ has a fixed point z_0 s.t. z_0 is an accumulation point of X, φ has stable orbits around z_0.

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$\varphi : X \to X$ has *stable orbits around a fixed point* z_0 if $\exists$ a fundamental family $(V_j)_j \subseteq X$ of connected compact neighbourhoods of z_0 s.t. $\varphi(V_j) \subseteq V_j$ $\forall j \in \mathbb{N}$.

Supercyclicity of $C_{w,\varphi}$ on spaces of continuous functions

Denjoy-Wolff theorem

If $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic is not the identity and not an automorphism with exactly one fixed point, then there is a unique (fixed) point $z_0 \in \overline{\mathbb{D}}$ such that $(\varphi^n)_n$ converges to z_0 uniformly on the compact subsets of $\mathbb{D}$.

Corollary

If $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic and has a fixed point, then $C_{w,\varphi}$ is not pointwise supercyclic on $(C(\mathbb{D}), \tau_p)$, neither on $E = H(\mathbb{D})$.

Corollary

If $X = \overline{\mathbb{D}}$ and $\varphi \in A(\mathbb{D})$, $C_{w,\varphi} : E \to E$ is never τ_p-supercyclic. Thus:

- $C_{w,\varphi}$ is never τ_p-supercyclic on $A(\mathbb{D})$.
- $C_{w,\varphi}$ is never τ_p-supercyclic on $Lip_\alpha(\mathbb{D})$, $0 < \alpha \leq 1$.

Section 2

Weak supercyclicity of composition operators on Fréchet spaces

Background:

- **Ansari and Bourdon (1997)**: If X is Banach and $T : X \to X$ is power bounded and supercyclic, then $(T^n(x))_n$ converges to 0 $\forall x \in X \Rightarrow$ *Isometries on Banach spaces are never supercyclic.*

- **Sanders (2005)**: Surjective isometries can be weakly supercyclic. $B : c_0(\mathbb{Z}) \to c_0(\mathbb{Z})$, $Be_j = e_{j-1}$, is a weakly supercyclic isometry.

Weak supercyclicity on locally convex spaces

Theorem (Beltrán-Menéu, J., Murillo-Arcila)

Let E be a locally convex space and $T : E \to E$ a weakly supercyclic operator satisfying $q \circ T \leq q$ for a continuous norm q of E. Then, $\sigma_p(T) \cap \partial\mathbb{D} = \emptyset$ and $\sigma_p(T') \cap \partial\mathbb{D} = \emptyset$. In particular, neither T nor T' have non zero fixed points.

As T is weakly supercyclic $\Leftrightarrow \alpha T$ is so:

Corollary

Let X be a Banach space. If $T : X \to X$ is a weakly supercyclic operator, then $\sigma_p(T) \subseteq B(0, \|T\|)$ and $\sigma_p(T^*) \subseteq B(0, \|T\|)$.

$\sigma_p(T^*)$ has at most 1 point (Peris (2001): l.c.s; Herrero (1991): on Hilbert).

Proof:

Enough to show $1 \notin \sigma_p(T)$ and $1 \notin \sigma_p(T')$ (T is weakly supercyclic $\Leftrightarrow \alpha T$ is for $\alpha \neq 0$).

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- $\underline{1 \notin \sigma_p(T'):}$ Let $U = \{e \in E : \ q(e) \leq 1\}$ and $K := (U^\circ, \omega^*)$,

$$U^\circ = \{u \in E' : |u(e)| \leq q(e) \text{ for all } e \in E\}.$$

$q \circ T \leq q \Rightarrow T'(U^\circ) \subseteq U^\circ$.

Proof:

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$$U^\circ = \{u \in E' : |u(e)| \leq q(e) \text{ for all } e \in E\}.$$

  $q \circ T \leq q \Rightarrow T'(U^\circ) \subseteq U^\circ$. There is a continuous injection $i : \ (E, \omega) \hookrightarrow (C(K), \tau_p)$ and $T = C_\varphi$, $\varphi = T' : U^\circ \mapsto U^\circ$. By Theorem ♣ (iv), T' does not have any fixed point.

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- $\underline{1 \notin \sigma_p(T)}$: Assume $T(e_0) = e_0$, $e_0 \in U$, $q(e_0) = 1$, and let

  $$F(e_0) := \{u \in U^\circ : \ u(e_0) = 1\}.$$

Proof:

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$$F(e_0) := \{u \in U^\circ : \ u(e_0) = 1\}.$$

  As $T'(F(e_0)) \subseteq F(e_0) \neq \emptyset$ and $F(e_0)$ is a ω^*-compact convex set, T' has a fixed point (Schauder-Tychonoff's fixed point theorem).

Background: Weak supercyclicity of C_φ on $H(\Omega)$

Let $\Omega \subseteq \mathbb{C}$ be a general planar domain.

- **Bernal-Gonzalez, Montes-Rodríguez (1995):** every simply connected domain admits an automorphism φ s.t. C_φ is hypercyclic.
- **Grosse-Erdmann and Mortini (2009)**: if $\Omega \subseteq \mathbb{C}$ is a non simply connected domain s.t. $\widehat{\mathbb{C}} \setminus \Omega$ has finitely many bounded components, $H(\Omega)$ does not support any hypercyclic C_φ (example: $\mathbb{C} \setminus \{0\}$).

Bès' problems (2014):

1. For which domains Ω does $H(\Omega)$ support a hypercyclic $C_{w,\varphi}$?
2. On $H(\Omega)$, Ω a planar domain not simply connected,

$$C_{w,\varphi} \text{ weakly supercyclic} \Leftrightarrow C_{w,\varphi} \text{ mixing?}$$

Weak supercyclicity on $H(\Omega)$

φ is *strongly runaway* if $\forall K \subseteq \Omega$ compact $\exists n_0 : \varphi^n(K) \cap K = \emptyset \ \forall n \geq n_0$.

Proposition (Beltrán-Menéu, J., Murillo-Arcila)

Let $\Omega \subseteq \mathbb{C}$, $\Omega \neq \mathbb{C}'$, be a domain and let $\varphi : \Omega \to \Omega$ be holomorphic. C_φ weakly supercyclic on $H(\Omega) \Rightarrow \varphi$ injective and strongly runaway.

Proof: Cases for φ on a hyperbolic plane domain (a domain $\neq \mathbb{C}, \mathbb{C}'$):
(1) φ strongly runaway (2) φ has a fixed point (3) $\varphi^{n_0} = \varphi$
(4) $\exists K$ with an acc. point s.t. $\varphi(K) \subseteq K \Rightarrow \|f \circ \varphi\|_K \leq \|f\|_K$.

Theorem Beltrán-Menéu, J., Murillo-Arcila

The spaces $H(\mathbb{D} \setminus \{0\})$ and $H(\mathbb{C} \setminus \{0\})$ admit no weakly supercyclic composition operators.

Sketch of the proof:

$\underline{H(\mathbb{D}\setminus\{0\}):}$ $\varphi:\mathbb{D}\setminus\{0\}\to\mathbb{D}\setminus\{0\}\Rightarrow\hat{\varphi}:\mathbb{D}\to\mathbb{D}$ holomorphic.

- If $\hat{\varphi}(0)\neq 0$, then $f\circ\varphi$ admits a holomorphic extension to $\{0\}$ $\forall f\in H(\mathbb{D}\setminus\{0\})$. $H(\mathbb{D})$ is closed in $H(\mathbb{D}\setminus\{0\})$.
- Assume $\widehat{\varphi}(0)=0$ and C_φ weakly supercyclic.
    - φ is strongly runaway and injective.
    - $\exists U\subseteq\mathbb{D}$, $0\in U$, $r>0$ s.t. $rU\subseteq U$ and $\widehat{\varphi}|_{D(a,r)}\sim g_a$, $g_a(z)=az$, $z\in\mathbb{D}$, $a\in\mathbb{C}$, $0<|a|<1$ (Koenigs).
    - C_{g_a} is not weakly supercyclic on $H(U\setminus\{0\})$:
      - Assume $\exists n_i\geq 1$ such that $\lim_i\lambda_i(f\circ(g_a)^{n_i})=1$.
      - Use the projections P_0 and P_{-1} on the Laurent development to get a contradiction.

Sketch of the proof:

$C_\varphi : H(\mathbb{C} \setminus \{0\}) \to H(\mathbb{C} \setminus \{0\})$:

- C_φ weakly supercyclic $\Rightarrow \varphi : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ injective.
- φ has the form $\varphi(z) = az$ or $\varphi(z) = \frac{a}{z}$, with $a \in \mathbb{C} \setminus \{0\}$.
  - $\varphi(z) = \frac{a}{z} \Rightarrow C_\varphi^2 = Id$.
  - $\varphi(z) = az$:
    - If $|a| \leq 1$: as in $\mathbb{D} \setminus \{0\}$.
    - If $|a| > 1$: as in $\mathbb{D} \setminus \{0\}$, but using projection P_1.

Section 3

$C_\varphi : C^m(\mathbb{R}) \hookrightarrow C^m(\mathbb{R})$, $m \in \mathbb{N} \cup \{\infty\}$

Supercyclicity and eigenvealues.

Theorem (Bayart-Matheron)

Let X be a separable lcHs and let $T \in \mathscr{L}(X)$ be supercyclic. Then either $\sigma_p(T') = \emptyset$ or $\sigma_p(T') = \{\lambda\}$, for some $\lambda \neq 0$. In the latter case, $\mathrm{Ker}(T' - \lambda)$ has dimension 1 and $\mathrm{Ker}(T' - \lambda)^n = \mathrm{Ker}(T' - \lambda)$ for all $n \in \mathbb{N}_0$. Moreover, there exists a (closed) T-invariant hyperplane $X_0 \subset X$ such that $T_0 := \lambda^{-1} T_{|X_0}$ is hypercyclic on X_0.

Proposition (Albanese, J., Mele)

$C_\varphi : C^m(\mathbb{R}) \to C^m(\mathbb{R})$ weakly supercyclic implies $\varphi'(x) \neq 0$ for all $x \in \mathbb{R}$ and φ has no fixed points.

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If $\varphi'(a) = 0$ then $C_\varphi(C^m(\mathbb{R})) \subseteq Ker\delta^1_a$. If $\varphi(a) = a$, δ^1_a and δ_a are two eigenvectors associated to $\varphi'(a)$ and a, and C_φ is not weakly supercyclic by Bayart-Matheron criterion.

Theorem (Albanese, J., Mele)

$C_\varphi : C^m(\mathbb{R}) \to C^m(\mathbb{R})$ is weakly supercyclic if and only if it is mixing.

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Proof

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- φ cannot have any convergence sequence $(\varphi^n(z))$, then φ is strongly runaway
- Kalmes characterization of mixing (weighted) composition operators on $C^m(\mathbb{R}^d)$.

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- Kalmes characterization of mixing (weighted) composition operators on $C(\mathbb{R}^d)$.

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