Compact Weighted Composition Operators on Spaces of Holomorphic Functions on Banach Spaces

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AIM

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We study when a weighted composition operator $C_{\psi,\varphi}$ is

- compact in the space of all bounded analytic functions $H^\infty(B)$,
- bounded, reflexive, Montel and (weakly) compact in the space of analytic functions of bounded type $H_b(B)$.

in terms of the properties of the *symbol* φ and *weight* ψ.

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If $U = B$ we denote $H(B)$ for the space of holomorphic functions $f : B \to \mathbb{C}$, endowed with the compact-open topology, denoted by τ_0.

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- $H_b(B)$ the subspace of $H(B)$ of functions of bounded type, which is Fréchet with the system of seminorms

$$\|f\|_r := \sup_{\|x\|<r} |f(x)|, \quad \text{where } 0 < r < 1.$$

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- $H^\infty(B)$ the subspace of $H(B)$ of bounded functions on B, which is Banach with the norm

$$\|f\|_\infty := \sup_{\|x\|<1} |f(x)|, \quad f \in H^\infty(B).$$

Definition

If $\psi\colon B \to \mathbb{C}$ and $\varphi\colon B \to B$ are holomorphic, the operator

$$C_{\psi,\varphi} : H(B) \to H(B), \quad f \mapsto \psi \cdot (f \circ \varphi)$$

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- Compactness of (weighted) composition operators defined on spaces of functions of one variable has been extensively studied (we mention works by different authors like Bonet, Contreras, Cowen, Díaz-Madrigal, Domański, Galindo, Hernández-Díaz, Lindström, MacCluer, Shapiro, Wikman...)

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- In spaces of holomorphic functions on infinite dimensional spaces the literature is much more scarce. Compactness of the composition operator C_φ defined on $H^\infty(B)$ was studied by Aron, Galindo and Lindström, and on $H_b(B)$ by Galindo, Lourenço and Moraes.

Continuity

Theorem

Assume φ, ψ are holomorphic and ψ is non-zero.

- $C_{\psi,\varphi}\colon H_b(B) \to H_b(B)$ *is continuous if and only if* ψ *and* φ *are of bounded type.*

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- $C_{\psi,\varphi}\colon H_b(B) \to H_b(B)$ *is continuous if and only if* ψ *and* φ *are of bounded type.*
- $C_{\psi,\varphi}\colon H^\infty(B) \to H^\infty(B)$ *is continuous if and only if* $\psi \in H^\infty(B)$.

Compactness on $H^\infty(B)$

We consider the point evaluation functional $\delta_x \colon H^\infty(B) \to \mathbb{C}$ on $H^\infty(B)$, defined as $\delta_x(f) = f(x)$ for $x \in B$. It belongs to the dual space $H^\infty(B)'$ and, moreover, $\|\delta_x\|_{H^\infty(B)'} = 1$ for every $x \in B$.

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Lemma

The operator $C_{\psi,\varphi} \colon H^\infty(B) \to H^\infty(B)$ is compact if and only if

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Proposition

Let $C_{\psi,\varphi} \colon H^\infty(B) \to H^\infty(B)$ be compact. Then $(\psi \cdot \varphi)(B)$ is relatively compact in X.

We extend for the infinite-dimensional case a result of Contreras and Díaz-Madrigal (which treated the case $B = \mathbb{D}$).

Theorem

Let $\psi \in H^\infty(B)$ and $\varphi\colon B \to B$ be holomorphic. Then the following conditions are equivalent:

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c) *$(\psi \cdot \varphi)(B)$ is relatively compact in X and one of the following properties hold:*
   - (i) *There is $0 < s < 1$ such that $\varphi(B) \subseteq sB$,*
   - (ii) $\lim_{r \to 1^-} \sup_{\|\varphi(x)\| > r} |\psi(x)| = 0.$

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- b) $\Rightarrow$ c) we proceed in a similar way to the corresponding result by Aron, Galindo and Lindström for composition operators assuming that neither (i) nor (ii) hold and proceed by contradiction.
- c) $\Rightarrow$ a) assume that $C_{\psi,\varphi}$ is not compact. Then there is a sequence $(f_n)_n \subset H^\infty(B)$ with $\|f_n\|_\infty \leq 1$ and $\varepsilon > 0$ with

$$\|\psi \cdot (f_n \circ \varphi) - \psi \cdot (f_m \circ \varphi)\|_\infty > \varepsilon,$$

for every $n < m$.

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for every $n < m$. We can select a set $\{x_{n,m} : n < m\} \subseteq B$:

$$|\psi(x_{n,m}) f_n(\varphi(x_{n,m})) - \psi(x_{n,m}) f_m(\varphi(x_{n,m}))| > \varepsilon, \text{ for } n < m.$$

Then the set $(\varphi(x_{n,m}))_{n<m}$ is not relatively compact in B.

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Then the set $(\varphi(x_{n,m}))_{n<m}$ is not relatively compact in B. Now, using the following technical lemma gives a contradiction with the fact that (f_n) is bounded in $H^\infty(B)$.

Sketch of proof II

Lemma

Let $\psi \in H^\infty(B)$, $\psi \neq 0$, and $\varphi : B \to B$ be holomorphic. Assume $(\psi \cdot \varphi)(B)$ is relatively compact in X and that one of the following holds:

(i) There is $0 < s < 1$ such that $\varphi(B) \subseteq sB$.

(ii) $\lim\limits_{r \to 1^-} \sup\limits_{\|\varphi(x)\| > r} |\psi(x)| = 0$.

Then, for each sequence (x_n) such that $(\varphi(x_n))$ is not relatively compact in B, there is a subsequence (x_{n_k}) such that

$$\lim_{k \to \infty} |\psi(x_{n_k})| = 0.$$

Definition

Let E, F be locally convex Hausdorff spaces and $T : E \to F$ a continuous linear operator. We say that:

- T is *bounded (compact, weakly compact)* if there is a 0-neighborhood U such that $T(U)$ is bounded (relatively compact, weakly relatively compact) in F.

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Proposition

Let E be a quasinormable Fréchet space. If $T : E \to E$ is a bounded linear operator which is also Montel (reflexive), then T is compact (weakly compact).

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The space $H_b(B)$ is quasinormable (Ansemil, Ponte).

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3. *$C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is bounded for some $\psi \in H_b(B)$, $\psi \neq 0$;*

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4. *$C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is bounded for every $\psi \in H_b(B)$.*

Corollary

Let B_N be the open unit ball of $\mathbb{C}^N$ with some norm. Let $\varphi : B_N \to B_N$ be holomorphic. The following are equivalent:

1. *there is $0 < s < 1$ such that $\varphi(B_N) \subseteq sB_N$;*
2. *$C_\varphi : H(B_N) \to H(B_N)$ is compact;*
3. *$C_{\psi,\varphi} : H(B_N) \to H(B_N)$ is compact for some $\psi \in H(B_N)$, $\psi \neq 0$;*
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4. *$C_{\psi,\varphi} : H(B_N) \to H(B_N)$ is compact for every $\psi \in H(B_N)$.*

Now, using the previous Proposition:

Corollary

Let $0 \neq \psi \in H_b(B)$ and $\varphi : B \to B$ be holomorphic. Then $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is compact if and only if

1. *$C_{\psi,\varphi}$ is Montel, and*
2. *There is $0 < s < 1$ such that $\varphi(B) \subseteq sB$.*

We denote by τ_0 the compact-open topology.

Lemma

Let $T : H_b(B) \to H_b(B)$ be a continuous linear operator such that it is also (τ_0, τ_0)-continuous. Consider:

(i) *$T : H_b(B) \to H_b(B)$ is Montel.*

(ii) *If $(f_i) \subset H_b(B)$ is bounded and $f_i \xrightarrow{\tau_0} 0$, then $Tf_i \to 0$ in $H_b(B)$.*

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Then (i) implies (ii). If, moreover, every compact set in $H(B)$ is sequentially compact, then (ii) implies (i).

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- Under other (technical) conditions (Cascales, Orihuela).
- In general it is not true:

Example

It is known that $\overline{B_{\ell'_\infty}}$ (the closed unit ball of ℓ'_∞) is not sequentially $\sigma(\ell'_\infty, \ell_\infty)$-compact. We deduce that there are compact sets in $H(B_{\ell_\infty})$ that are not sequentially compact.

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- *If $C_{\psi,\varphi}$ is (reflexive) Montel, then for each $0 < r < 1$ the set A_r is (weakly) relatively compact in $H_b(B)'$.*

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- *Conversely, if A_r is relatively compact in $H_b(B)'$ for each $0 < r < 1$, then $C_{\psi,\varphi}$ is Montel.*

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- *Conversely, if A_r is relatively compact in $H_b(B)'$ for each $0 < r < 1$, then $C_{\psi,\varphi}$ is Montel.*

Corollary

If $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is (reflexive) Montel, then the set $(\psi \cdot \varphi)(rB)$ is (weakly) relatively compact in X for every $0 < r < 1$.

Theorem

Let ψ and φ be holomorphic of bounded type. We have:

1. $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ *is Montel if and only if* $(\psi \cdot \varphi)(rB)$ *is relatively compact in* X *for every* $0 < r < 1$*;*

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2. *If* $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ *is reflexive, then* $(\psi \cdot \varphi)(rB)$ *is weakly relatively compact in* X *for every* $0 < r < 1$*.*
   Moreover, if X *has the Schur property and* $(\psi \cdot \varphi)(rB)$ *is weakly relatively compact in* X*, then* $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ *is reflexive.*

Theorem 1

Let ψ and φ be of bounded type. T.F.A.E.:

(i) $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is compact;

(ii) The following two conditions hold:

(a) $(\psi \cdot \varphi)(rB)$ is relatively compact in X for every $0 < r < 1$.

(b) There is $0 < s < 1$ such that $\varphi(B) \subseteq sB$.

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Theorem 2

Let ψ and φ be holomorphic of bounded type:

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Then (i) $\Rightarrow$ (ii) and, if X has the Schur property, (ii) $\Rightarrow$ (i).

Corollary

Let φ *be holomorphic. Then,* $C_\varphi : H_b(B) \to H_b(B)$ *is compact if and only if there is* $0 < s < 1$ *such that* $\varphi(B) \subseteq sB$ *and for each* $0 < r < 1$ *the set* $\varphi(rB)$ *is relatively compact in* X*.*

Corollary

Let φ be holomorphic. Then, $C_\varphi : H_b(B) \to H_b(B)$ is compact if and only if there is $0 < s < 1$ such that $\varphi(B) \subseteq sB$ and for each $0 < r < 1$ the set $\varphi(rB)$ is relatively compact in X.

Open problem

We do not know if there is $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ Montel (compact) so that $C_\varphi : H_b(B) \to H_b(B)$ is not Montel (compact).

However, we can see that this is not the case when φ is holomorphic of bounded type and open.

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Proposition

Let $\psi \in H_b(B)$ and $\varphi : B \to B$ be holomorphic of bounded type and open. If $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is Montel then X is finite dimensional. Consequently, $C_\varphi : H_b(B) \to H_b(B)$ is also Montel.

Example

Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi \in H^\infty(\mathbb{D})$ defined by

$$\varphi(z) = \frac{1+z}{2} \quad \text{and} \quad \psi(z) = 1 - z.$$

Then $C_{\psi,\varphi} : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ is compact, but
$C_\varphi = C_{1,\varphi} : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ is not compact.

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Example

Assume that X is a Banach space of infinite dimension.
Consider $\varphi : B \to B$ defined by $\varphi(x) = \frac{1}{2}x$.

1. The operator $C_\varphi : H^\infty(B) \to H^\infty(B)$ is continuous but it is not compact, therefore it is bounded but it is not Montel.

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2. The operator $C_\varphi : H_b(B) \to H_b(B)$ is bounded but it is not Montel.

Example

Let $\varphi : B_{c_0} \to B_{c_0}$ defined by

$$\varphi(x) = \frac{1}{2}\left(x_n^n\right).$$

Then the composition operator C_φ is compact in $H_b(B_{c_0})$, but it is not compact in $H^\infty(B_{c_0})$.

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Example

Let $\varphi : B_{c_0} \to B_{c_0}$ defined by $\varphi(x) = (x_n^n)$. The composition operator $C_\varphi : H_b(B_{c_0}) \to H_b(B_{c_0})$ is Montel, but not bounded and hence, not compact either.