Quantitative Runge type approximation theorems for kernels of partial differential operators

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Introduction

Runge's Approximation Theorem

For $Y \subseteq X \subseteq \mathbb{C}$ open the following are equivalent.

i) For every $g \in \mathscr{H}(Y)$, for every compact $K \subseteq Y$, and for every $\varepsilon > 0$ there is $f \in \mathscr{H}(X)$ such that

$$\varepsilon > \sup_{z \in K} |f(z) - g(z)| =: \|f - g\|_{0,K},$$

i.e. $r : \mathscr{H}(X) \to \mathscr{H}(Y), f \mapsto f_{|Y}$ has dense range when $\mathscr{H}(Y)$ is equipped with the topology of local uniform convergence.

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where $P \in \mathbb{C}[X_1, \ldots, X_d]$ is elliptic, $D = -i(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$.

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$$\forall\, r \in \mathbb{N}_0, K \subseteq X \text{ compact} : \|f\|_{r,K} = \max_{|\alpha| \le r, x \in K} |\partial^\alpha f(x)|,\ f \in \mathscr{E}_P(X).$$

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(Examples: $P(\xi_1, \xi_2) = \frac{i}{2}(\xi_1 + i\xi_2) \Rightarrow P(D) = \partial_{\bar{z}}$; $P(\xi) = -|\xi|^2 \Rightarrow P(D) = \Delta$)

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P non-elliptic $\Rightarrow \exists\, \xi \in \mathbb{R}^d\backslash\{0\} : P_m(\xi) = 0$ ($:\Leftrightarrow$ ξ is *characteristic* for P)
P_m is homogeneous (of degree m) $\Rightarrow$ w.l.o.g. $|\xi| = 1$ and by an orthogonal change of variables $\xi = e_1 = (1, 0, \ldots, 0)$.

Runge type theorems for certain non-elliptic partial differential operators

For P we define $\check{P}(\xi) := P(-\xi)$ $(\Rightarrow \check{P}_m(\xi) = (-1)^m P_m(\xi))$. Moreover, let $H := \{x \in \mathbb{R}^d;\, 0 = x_1\, (= \langle x, e_1 \rangle)\}$

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Theorem [4]

Let P be s.th. e_1 is characteristic for P. Assume that the following is true:

$$\forall\, \varepsilon > 0\, \exists\, v \in \mathscr{E}_{\check{P}}(\mathbb{R}^d) : [-\varepsilon/2, \varepsilon/2] \times \mathbb{R}^{d-1} \subseteq \operatorname{supp} v \subseteq [-\varepsilon, \varepsilon] \times \mathbb{R}^{d-1},$$
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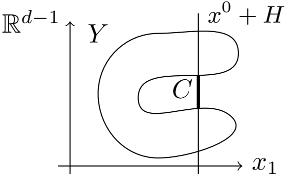

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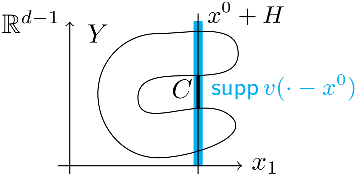

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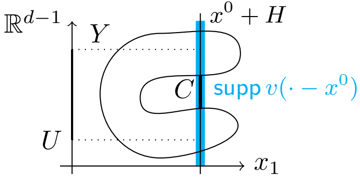

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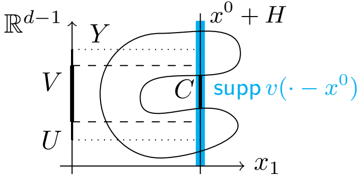

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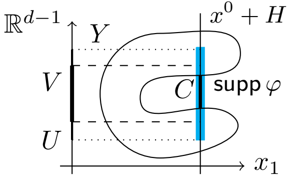

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$\chi = 1$ in a neighborhood of $\overline{V} \subseteq U$
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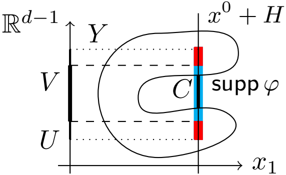

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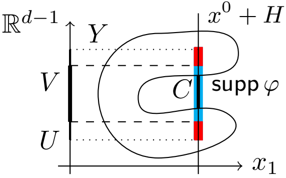

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$\forall\, u \in \mathscr{D}'_P(X) : \langle r_{\mathscr{D}'}(u), \check{P}(D)\varphi\rangle = \langle u, \check{P}(D)\varphi\rangle = \langle P(D)u, \varphi\rangle = 0$ but $\exists\, v \in \mathscr{D}'_P(Y) : \langle v, \check{P}(D)\varphi\rangle \neq 0$ which gives a contradiction ☺

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$$\forall \varepsilon > 0 \, \exists \, v \in \mathscr{E}_{\check{P}}(\mathbb{R}^d) : \, [-\varepsilon/2, \varepsilon/2] \times \mathbb{R}^{d-1} \subseteq \operatorname{supp} v \subseteq [-\varepsilon, \varepsilon] \times \mathbb{R}^{d-1},$$
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- P *acts along* W $:\Leftrightarrow \forall\, \xi \in \mathbb{R}^d : P(\xi) = P(\pi_W \xi)$ (π_W orthogonal projection onto W)
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$$R \in \mathbb{C}[X_1, \dots, X_{d-1}], \deg R < m : P(\xi) := R(\xi_1, \dots, \xi_{d-1}) + \tilde{P}(\xi_k, \dots, \xi_d)$$

satisfies the above hypothesis.

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satisfies the above hypothesis.

Concrete: $P(D) = \partial_t - \Delta_x$; $P(D) = i\partial_t + \Delta_x$; $\tilde{P}(D) = \sum_{j=1}^{d} \alpha_j \partial_j + \gamma$, $\alpha_j \in \mathbb{R}$.

Sufficient condition for

$$\mathscr{E}_P(Y) = \overline{r_{\mathscr{E}}\left(\mathscr{E}_P(X)\right)} \text{ resp. } \mathscr{D}'_P(Y) = \overline{r_{\mathscr{D}'}\left(\mathscr{D}'_P(X)\right)}\,?$$

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$$E = \overline{F} \Leftrightarrow \forall\,\psi \in E' : \big(\psi_{|F} = 0 \Rightarrow \psi = 0\big)\,;$$

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having to deal with the closures is a problem!

For $P \neq 0, X \subseteq \mathbb{R}^d$ open, tfae (Floret: $(i) \Leftrightarrow (ii)$)

(i) $\overline{\check{P}(D)(\mathscr{D}(X))} = \check{P}(D)(\mathscr{D}(X))$.

(ii) $\overline{\check{P}(D)(\mathscr{E}'(X))} = \check{P}(D)(\mathscr{E}'(X))$.

(iii) $P(D) : \mathscr{E}(X) \to \mathscr{E}(X)$ is surjective.

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Theorem [2]

Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be elliptic along the subspace W, $Y \subseteq X \subseteq \mathbb{R}^d$ be open, X be P-convex for supports. Then, $(i) \Rightarrow (ii)$, where

(i) $\nexists\, x \in \mathbb{R}^d :\ X$ contains compact connected component of $\left(\mathbb{R}^d \backslash Y\right) \cap (x + W)$.

(ii) $\mathscr{E}_P(Y) = \overline{r_{\mathscr{E}}\left(\mathscr{E}_P(X)\right)}$ and/or $\mathscr{D}'_P(Y) = \overline{r_{\mathscr{D}'}\left(\mathscr{D}'_P(X)\right)}$.

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If $d = 2$ (or $W = \mathbb{R}^d$) we also have $(ii) \Rightarrow (i)$.

Corollary [2]

Let $P(D) = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$, $Y \subseteq X \subseteq \mathbb{R}^2$ be open, X be P-convex for supports. Tfae:

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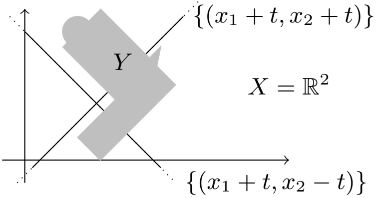

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Applicable to $P(D) = \partial_t - \Delta_x$ and $P(D) = i\partial_t + \Delta_x$;

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Applicable to $P(D) = \partial_t - \Delta_x$ and $P(D) = i\partial_t + \Delta_x$;
in particular for

$$Y = (\alpha_1, \beta_1) \times G_1 \subseteq (\alpha_2, \beta_2) \times G_2 = X;\, G_j \subseteq \mathbb{R}^{d-1}\text{open}, j = 1, 2 :$$

Dense range iff G_2 does not contain a co.co.co. of $\mathbb{R}^{d-1} \backslash G_1$.

Quantitative Runge type approximation theorems

Rüland, Salo (2019): $Z \subseteq X \subseteq \mathbb{R}^d$ open, bounded Lipschitz domains such that $\overline{Z} \subseteq X$, $X \backslash \overline{Z}$ connnected $\left(\Rightarrow X\right.$ does not contain bounded co.co. of $\left.\mathbb{R}^d \backslash \overline{Z}\right)$

Let $(a_{j,k})_{1 \leq j,k \leq d} \in W^{1,\infty}(X)^{d \times d}$ be a real, symmetric matrix function, $c \in L^\infty(X)$ such that

$$Lu = \sum_{j,k=1}^{d} \partial_k \left(a_{j,k} \partial_j u\right) + c\, u, u \in H^1(X)$$

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is uniformly elliptic (+ some additional technical assumptions).
Let Y be a bounded Lipschitz domain with $\overline{Z} \subseteq Y, \overline{Y} \subseteq X$. Then there are $C > 0, s > 1$ such that

$$\forall\, g \in H^1(Y), Lg = 0 \text{ in } Y\, \forall\, \varepsilon \in (0,1)\, \exists f \in H^1(X), Lf = 0 \text{ in } X :$$
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(quantitative Runge type approximation)

Rüland, Salo (2019): $Z \subseteq X \subseteq \mathbb{R}^d$ open, bounded Lipschitz domains such that $\overline{Z} \subseteq X$, $X \backslash \overline{Z}$ connnected $\left(\Rightarrow X\right.$ does not contain bounded co.co. of $\left.\mathbb{R}^d \backslash \overline{Z}\right)$

Let $(a_{j,k})_{1 \leq j,k \leq d} \in W^{1,\infty}(X)^{d \times d}$ be a real, symmetric matrix function, $c \in L^\infty(X)$ such that

$$Lu = \sum_{j,k=1}^{d} \partial_k \left(a_{j,k} \partial_j u\right) + c\, u, u \in H^1(X)$$

is uniformly elliptic (+ some additional technical assumptions).
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(quantitative Runge type approximation)

Second objective: Generalization in the context of constant coefficient partial differential operators

Recall the linear topological invariant (Ω) of Vogt, Wagner:

A Fréchet space E (with increasing fundamental sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of seminorms) satisfies (Ω) iff

$$\forall\, k \in \mathbb{N}\, \exists\, l \geq k\, \forall\, m \geq l\, \exists\, C > 0, s > 1$$

$$\forall\, \varepsilon \in (0,1)\, \forall\, f \in E\, \exists\, h \in E: \; \|f - h\|_k \leq \varepsilon \|f\|_l \text{ and } \|h\|_m \leq \frac{C}{\varepsilon^s}\|f\|_l.$$

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(Ω) for $\mathscr{E}_P(X)$ ($K\Subset X :\Leftrightarrow K\subseteq X$ compact):

$$\forall\, K\Subset X, k\in\mathbb{N}_0\,\exists\, L\Subset X, K\subseteq L, l\geq k\,\forall\, M\Subset X, m\geq l\,\exists\, C>0, s>1$$
$$\forall\,\varepsilon\in(0,1)\,\forall\, f\in\mathscr{E}_P(X)\,\exists\, h\in\mathscr{E}_P(X):$$
$$\|f-h\|_{k,K}\leq\varepsilon\|f\|_{l,L} \text{ and } \|h\|_{m,M}\leq\frac{C}{\varepsilon^s}\|f\|_{l,L}.$$

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Assume $\mathscr{E}_P(Y)=\overline{r_{\mathscr{E}}(\mathscr{E}_P(X))}$, $\mathscr{E}_P(X)$ has (Ω) <u>and</u> that for $K\Subset Y$ one can choose $L\Subset Y$:

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$$\|g-h\|_{k,K}\leq\varepsilon\|g\|_{l,L} \text{ and } \|h\|_{m,M}\leq\frac{4^s 2C}{\varepsilon^s}\|g\|_{l,L}.$$

Theorem [1], [3]

Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ and let $X \subseteq \mathbb{R}^d$ be open and P-convex for supports. Then, $\mathscr{E}_P(X)$ has (Ω) in each of the following cases.

(i) $d = 2$.

(ii) X convex (for hypoelliptic P this is due to Petzsche).

(iii) P is elliptic along a subspace W (for $W = \mathbb{R}^d$ this is due to Vogt).

(iv) $\{\xi \in \mathbb{R}^d; P_m(\xi) = 0\} = \mathrm{span}\{e_1\}$ and P is semi-elliptic; covers polynomials like $P(\xi) = i\xi_1^r - \tilde{P}(\xi_2, \ldots, \xi_d)$ with $\tilde{P}$ elliptic of degree $m > r$ and real coefficients in its principal part; concrete $P(D) = \partial_t - \Delta_x$.

Theorem [2]

Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be elliptic along the subspace W. Let $Y \subseteq X \subseteq \mathbb{R}^d$ be open, X be P-convex for supports such that

$$\nexists\, x \in \mathbb{R}^d : X \text{ contains a co.co.co. of } \left(\mathbb{R}^d \backslash Y\right) \cap (x + W).$$

Moreover, let $\overline{\mathrm{int}(K)} = K \Subset Y$ be such that

$$\nexists\, x \in \mathbb{R}^d : X \text{ contains a bounded co.co. of } (\mathbb{R}^d \backslash K) \cap (x + W).$$

Then,

$$\forall\, L \Subset Y, K \Subset \mathrm{int}(L)\, \forall\, M \Subset X\, \exists\, s > 1\, \forall\, k, m \in \mathbb{N}_0\, \exists\, C > 0\, \forall\, \varepsilon \in (0,1)$$
$$\forall g \in \mathscr{E}_P(Y)\, \exists h \in \mathscr{E}_P(X) : \|g - h\|_{k,K} \leq \varepsilon \|g\|_{k+1,L} \text{ and } \|h\|_{m,M} \leq \frac{C}{\varepsilon^s} \|g\|_{k+1,L}.$$

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Applicable to $W = \mathbb{R}^d \Rightarrow P$ elliptic

Theorem [2]

Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be elliptic . Let $Y \subseteq X \subseteq \mathbb{R}^d$ be open such that

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This generalizes Rüland, Salo; due to Petzsche in case of $d = 2$ and $P(D) = \frac{1}{2}(\partial_1 + i\partial_2)$

Corollary [2]

Let $[t_1, t_2], [a, b] \subseteq \mathbb{R}$, $\delta > 0$. Then,

$$\forall\, M \Subset \mathbb{R}^2\, \exists\, s > 1\, \forall\, k, m \in \mathbb{N}_0\, \exists\, C > 0\, \forall\, \varepsilon \in (0, 1)$$
$$\forall\, g \in \mathscr{E}\left((t_1 - \delta, t_2 + \delta) \times (a - \delta, b + \delta)\right), \partial_t^2 g - \partial_x^2 g = 0\, \exists\, h \in \mathscr{E}(\mathbb{R}^2) :$$
$$\partial_t^2 h - \partial_x^2 h = 0 \text{ in } \mathbb{R}^2, \|g - h\|_{k,[t_1,t_2]\times[a,b]} \leq \varepsilon \|g\|_{k+1,[t_1-\delta/2,t_2+\delta/2]\times[a-\delta/2,b+\delta/2]}$$
$$\text{and } \|h\|_{m,M} \leq \frac{C}{\varepsilon^s} \|g\|_{k+1,[t_1-\delta/2,t_2+\delta/2]\times[a-\delta/2,b+\delta/2]}.$$

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Let $\tilde{P} \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ be elliptic, of degree m, with real coefficients in its principal part, $r < m$. Set $P(\xi) = i\xi_1^r - \tilde{P}(\xi_2, \ldots, \xi_d)$. Moreover, let

(i) $G \subseteq \mathbb{R}^{d-1}$ open
(ii) $K \Subset G$, ∂K continuous
(iii) G contains no bounded co.co. of $\mathbb{R}^d \backslash K$
(iv) $D \subseteq \mathbb{R}^{d-1}, I \subseteq \mathbb{R}$ open, $\overline{D} \subseteq G$
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Let $[t_1, t_2], [a, b] \subseteq \mathbb{R}$, $\delta > 0$. Then,

$$\forall\, M \Subset \mathbb{R}^2 \,\exists\, s > 1 \,\forall\, k, m \in \mathbb{N}_0 \,\exists\, C > 0 \,\forall\, \varepsilon \in (0,1)$$
$$\forall\, g \in \mathscr{E}\left((t_1 - \delta, t_2 + \delta) \times (a - \delta, b + \delta)\right), \partial_t^2 g - \partial_x^2 g = 0 \,\exists\, h \in \mathscr{E}(\mathbb{R}^2) :$$
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Then, $\forall\, M \Subset \mathbb{R} \times G \,\exists\, s > 1, C > 0 \,\forall\, g \in \mathscr{E}_P(I \times D), \varepsilon > 0 \,\exists\, h \in \mathscr{E}_P(\mathbb{R} \times G)$:

$$\|g - h\|_{[t_1,t_2]\times K} \leq \varepsilon \|g\|_{[t_1-\delta,t_2+\delta]\times L} \text{ and } \|h\|_M \leq \frac{C}{\varepsilon^s} \|g\|_{[t_1-\delta,t_2+\delta]\times L}.$$

For $r > 0$ set $B(0,r) = \{x \in \mathbb{R}^d; |x| < r\}$ and $B[0,r] = \{x \in \mathbb{R}^d; |x| \leq r\}$

Theorem [2]

Let $P \in \mathbb{C}[X_1, \ldots, X_d] \backslash \{0\}$ and $r_1, r_2, \delta > 0$ with $r_1 + \delta < r_2$. Then, for every $r_3 > r_2$ there is $s > 1$ such that for every $k, m \in \mathbb{N}_0$ there is $C > 0$ such that

$$\forall\, g \in \mathscr{E}_P(B(0,r_2)), \varepsilon \in (0,1)\, \exists\, h \in \mathscr{E}_P(\mathbb{R}^d) : \|g - h\|_{k,B[0,r_1]} \leq \varepsilon \|g\|_{k+1,B[0,r_1+\delta]}$$
$$\text{and } \|h\|_{m,B[0,r_3]} \leq \frac{C}{\varepsilon^s} \|g\|_{k+1,B[0,r_1+\delta]}.$$

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