Dynamics of skew-products of differential operators

Félix Martínez-Giménez

Instituto Universitario Matemática Pura y Aplicada
Universitat Politècnica de València

Joint work with
H. Méndez, A. Peris y F. Rodenas

Workshop on Functional and Complex Analysis
Valladolid (2022)

Context

Definition: *Skew-product* of an operator

A compact metric space
$f : A \to A$ continuous
$h : A \to \mathbb{C}$ continuous

X complex separable Fréchet space
$T : X \to X$ continuous and linear operator

$$P : A \times X \to A \times X$$
$$(a, x) \mapsto (f(a), h(a)Tx)$$

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Additionally

1. μ is a probability measure on $(A, \mathcal{A})$ ($\mu : \mathcal{A} \to \mathbb{R}^+,\ \mu(A) = 1$)
2. f is ergodic with respect ot μ on $(A, \mathcal{A})$
   - f is μ-invariant ($\mu(f^{-1}(\Gamma)) = \mu(\Gamma)$ for all $\Gamma \in \mathcal{A}$)
   - If $f^{-1}(\Gamma) = \Gamma$, $\Gamma \in \mathcal{A}$ then $\mu(\Gamma) \in \{0, 1\}$
3. μ has full support ($\mu(U) > 0$ for all U open and non-empty)

Aim

To study dynamical properties like transitivity, mixing and chaos in the sense of Devaney for skew-products of operators, in particular for skew-products of differential operators on $H(\mathbb{C})$

Recall that an endomorphism f on a topological space

- is topologically transitive if for any U, V non-empty open sets

$$\exists n \in \mathbb{N} \text{ s.t. } f^n(U) \cap V \neq \emptyset$$

- is topologically mixing if for any U, V non-empty open sets

$$\exists N \in \mathbb{N} \text{ s.t. } f^n(U) \cap V \neq \emptyset \; \forall n \geq N$$

- is chaotic in the sense of Devaney if it is topological transitive and it admits a dense set of periodic points

Previous works and motivation

Bayart-Costakis-Hadjiloucas (2+3 (2008), 1+2+3 (2010))

- Provide sufficient conditions for transitivity of skew-products of operators defined on Banach spaces
- Study transitivity of skew-products of unilateral weighted backward shifts on ℓ^p
- Study transitivity of skew-products of composition operators on $H^2(\mathbb{D})$ associated to $\phi \in \mathrm{Aut}(\mathbb{D})$
- For Fréchet spaces they proved that skew-products of translations and the differentiation operators on $H(\mathbb{C})$ are transitive

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Godefroy-Shapiro (1991)

Suppose that $T : H(\mathbb{C}) \to H(\mathbb{C})$, $T \neq \lambda I$, is an operator that commutes with the differentiation operator D, that is, $T \circ D = D \circ T$. Then T is mixing and chaotic ($H(\mathbb{C})$ with topology of uniform convergence on compact sets).

Theorem

Let A be a compact metric space, $f : A \to A$ a continuous map, μ an ergodic probability measure on A for f giving non-zero measure to every non-empty open set and $h : A \to \mathbb{C}$ a continuous function.
Let $T : H(\mathbb{C}) \to H(\mathbb{C})$, $T \neq \lambda I$, be an operator that commutes with the differentiation operator D.
Suppose that $\gamma := \int_A log\,|h|\, d\mu$ is finite and consider the skew-product

$$
\begin{aligned}
P : A \times H(\mathbb{C}) &\to A \times H(\mathbb{C}) \\
(a\,,u) &\longmapsto (f(a), h(a)Tu)
\end{aligned}
$$

i) P is transitive

ii) P is chaotic if f is chaotic and $|h| > 0$

Tools

Commutant of D (Godefroy-Shapiro)

If T commutes with D then $T = \varphi(D)$ where $\varphi(z)$ is an entire function of exponential type. If $\varphi(z) = \sum_{n\geq 0} a_n z^n$, then for $\lambda \in \mathbb{C}$ we have

$$T \exp(\lambda z) = \sum_{n\geq 0} a_n D^n \exp(\lambda z) = \sum_{n\geq 0} a_n \lambda^n \exp(\lambda z) = \varphi(\lambda) \exp(\lambda z)$$

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Birkhoff Ergodic Theorem

For every $\phi \in L^1(\mu)$ and for μ-almost every $a \in A$ we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \phi(f^n a) \xrightarrow{N\to\infty} \int_A \phi d\mu$$

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Density of exponentials on $H(\mathbb{C})$

Let $\Lambda \subset \mathbb{C}$ be a set with an accumulation point. Then the set

$$\mathrm{span}\{\exp(\lambda z) \ : \ \lambda \in \Lambda\}$$

is dense in $H(\mathbb{C})$.

Notation

Iterations of P

$$P(a,u) = (f(a), h(a)Tu)$$
$$P^2(a,u) = P(f(a), h(a)Tu) = (f^2(a), h(f(a))h(a)T^2u)$$
$$P^n(a,u) = (f^n(a), h_n(a)T^nu), \text{ where } h_n(a) := h(f^{n-1}(a))\dots h(f(a))h(a)$$

The base and the fibre of a skew-product

For $a \in A$ (the base),
we have a sequence of operators $(T_{a,n})$ (the fibre) given by

$$T_{a,n} : H(\mathbb{C}) \to H(\mathbb{C}), \; T_{a,n} := h_n(a)T^n, \; n \geq 1$$

$$\begin{aligned} \mathrm{Orb}(P,(a,u)) &= \{P^n(a,u) \; : \; n \geq 0\} \\ &= \{(f^n(a), T_{a,n}(u)) \; : \; n \geq 0\} \end{aligned}$$

Sketch of proof (transitivity of P)

- $a, c \in A$, $\varepsilon > 0$, U, V open $\neq \emptyset \Rightarrow B(a,\varepsilon) \times U$ and $B(c,\varepsilon) \times V$
- **GOAL!**
  Find $(b,u) \in B(a,\varepsilon) \times U$ such that $P^n(b,u) \in B(c,\varepsilon) \times V$ for some $n \in \mathbb{N}$
  That is, $f^n(b) \in B(c,\varepsilon)$ and $T_{b,n}(u) = h_n(b)T^n u \in V$, but
  $T_{b,n}(u) = h_n(b)T^n u = h_n(b)\varphi(D)^n u = h_n(b)(\sum_{k\geq 0} a_k D^k)^n u$

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  $T_{b,n}(u) = h_n(b)T^n u = h_n(b)\varphi(D)^n u = h_n(b)(\sum_{k\geq 0} a_k D^k)^n u$
- Take $b \in A_1 \cap A_2 \cap B(a,\varepsilon)$

$$A_1 := \left\{ b \in A \ : \ \frac{1}{n}\sum_{j=0}^{n-1} \chi_{B(c,\varepsilon)}(f^j(b)) \xrightarrow{n\to\infty} \mu(B(c,\varepsilon)) \right\}$$

$$A_2 := \left\{ b \in A \ : \ \frac{1}{n}\sum_{j=0}^{n-1} \log\left|h(f^j(b))\right| \xrightarrow{n\to\infty} \int_A \log|h|d\mu \right\}$$

$b \in A_2$ means $\delta > 0$, $\exists N$ such that if $n \geq N$

$$\left| \frac{1}{n}\sum_{j=0}^{n-1} \log\left|h(f^j(b))\right| - \gamma \right| < \delta$$

$$\exp\left(n(\gamma-\delta)\right) < |h_n(b)| < \exp\left(n(\gamma+\delta)\right)$$

- Recall that $T^n \exp(\lambda z) = \varphi(\lambda)^n \exp(\lambda z)$

- The sets $\begin{array}{l} \text{span}\{\exp(\lambda z) \ : \ |\varphi(\lambda)| < \exp(-\gamma)\} \\ \text{span}\{\exp(\lambda z) \ : \ |\varphi(\lambda)| > \exp(-\gamma)\} \end{array}$ are dense in $H(\mathbb{C})$

$$\begin{array}{c} u \in U \cap \text{span}\{\exp(\lambda z) \ : \ |\varphi(\lambda)| < \exp(-\gamma)\} \\ u = \sum_{k=1}^{m} a_k \exp(\lambda_k z) \ : \ |\varphi(\lambda_k)| < \exp(-\gamma), \forall k \\ v \in V \cap \text{span}\{\exp(\lambda z) \ : \ |\varphi(\lambda)| > \exp(-\gamma)\} \\ v = \sum_{k=1}^{m} b_k \exp(\mu_k z) \ : \ |\varphi(\mu_k)| > \exp(-\gamma), \forall k \end{array}$$

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- Define $u_n = \displaystyle\sum_{k=1}^{m} b_k \frac{1}{h_n(b)\varphi(\mu_k)^n} \exp(\mu_k z)$, $n \geq 1$

1. $u_n \xrightarrow{n\to\infty} 0$
2. $T_{b,n} u_n = v$ for all $n \geq 1$

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1. $u_n \xrightarrow{n\to\infty} 0$
2. $T_{b,n} u_n = v$ for all $n \geq 1$

- Take the sequence $(u + u_n)_n$
  1. $u + u_n \in U$ for $n >>$
  2. $T_{b,n} u \xrightarrow{n\to\infty} 0$
  3. $T_{b,n}(u + u_n) = T_{b,n} u + T_{b,n} u_n = T_{b,n} u + v \in V$ for $n >>$
- Therefore $(T_{b,n})_n$ is transitive (even mixing!)

Observe also that $(f^n(b))_n$ must meet $B(c,\varepsilon)$ since $b \in A_1$. Recall that

$$A_1 := \left\{ b \in A \ : \ \frac{1}{n} \sum_{j=0}^{n-1} \chi_{B(c,\varepsilon)}(f^j(b)) \xrightarrow{n\to\infty} \mu(B(c,\varepsilon)) \right\}$$

Sketch of proof (P is chaotic if f is chaotic)

- Take $a \in A$ n-periodic for f
  $P^n(a,u) = (f^n(a), h_n(a)T^n u) = (a, h_n(a)T^n u)$
  $P^{nk}(a,u) = (a, h_n(a)^k T^{nk} u),\ k \geq 1$
- If $u = \exp(\lambda z)$ we have
  $P^{nk}(a, \exp(\lambda z)) = (a, (h_n(a)\varphi(\lambda)^n)^k \exp(\lambda z)),\ k \geq 1$
- Idea: take enough exponentials satisfying $(h_n(a)\varphi(\lambda)^n)^k = 1$

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- Idea: take enough exponentials satisfying $(h_n(a)\varphi(\lambda)^n)^k = 1$
- Since $|h| > 0$ we have $h_n(a) \neq 0$. Suppose $h_n(a) \in \mathbb{R}$ (if not rotate!).
  Take $\Lambda := \{|h_n(a)|^{-1/n} exp(\alpha\pi i),\ \alpha \in \mathbb{Q}\}$
- All vectors in $\mathrm{span}\{exp(\lambda z)\ :\ \varphi(\lambda) \in \Lambda\}$ are periodic for $(T_{a,n})_{n\geq 0}$ and the set is dense in $H(\mathbb{C})$
- The following is a dense set of periodic points for P

$$\bigcup_{a\ f\text{-periodic}} \{(a,u)\ : u \in \mathrm{span}\{exp(\lambda z)\ :\ \varphi(\lambda) \in \Lambda\}\}$$

Note that a single periodic point of f will suffice for the fibre $(T_{a,n})_{n\geq 0}$ to have a dense set of periodic points (for example a single fixed point)

Toplogical dynamics approach with no ergodicity

Theorem

Let A be a compact metric space, $f : A \to A$ a continuous map and $h : A \to \mathbb{C}$ a continuous function with $|h| > 0$.
Let $T : H(\mathbb{C}) \to H(\mathbb{C})$, $T \neq \lambda I$, be an operator that commutes with the differentiation operator D.
Consider the skew-product

$$\begin{aligned} P : A \times X &\to A \times X \\ (a, x) &\mapsto (f(a), h(a)Tx) \end{aligned}$$

i) P is transitive if f is transitive
ii) P is chaotic if f is chaotic

Referencias

F. Bayart, G. Costakis, and D. Hadjiloucas, *Topologically transitive skew-products of operators*, Ergodic Theory Dynam. Systems **30** (2010), no. 1, 33–49.

G. Costakis and D. Hadjiloucas, *Topologically transitive skew-products of backward shift operators and hypercyclicity*, Proc. Amer. Math. Soc. **136** (2008), no. 3, 937–946.

G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98** (1991), no. 2, 229–269.