On the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$

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Multipliers and convolutors

Classicaly

$$\mathcal{O}_M(\mathbb{R}^N) = \left\{ f \in C^\infty(\mathbb{R}^N) : \ \forall \alpha \, \exists k : (1+|x|^2)^{-k} \partial^\alpha f(x) \in C_0 \right\}$$

$$\mathcal{O}_C(\mathbb{R}^N) = \left\{ f \in C^\infty(\mathbb{R}^N) : \ \exists k \, \forall \alpha : (1+|x|^2)^{-k} \partial^\alpha f(x) \in C_0 \right\}$$

Gelfand-Shilov type spaces

$$\mathcal{Z}_{[\mathcal{V}]}^{[w]}(\mathbb{R}^N)$$

$$\mathcal{O}_C^{[M_p],[A_p]}(\mathbb{R}^N)$$

Weight functions (in the sense of BMT) 1/2

Definition

A *non-quasianalytic weight function* is a continuous increasing function $\omega\colon [0,\infty) \to [0,\infty)$ satisfying the following properties:

(α) $\exists K \geq 1$ such that $\omega(2t) \leq K(1+\omega(t))$, $\forall t \geq 0$;

(β) $\int_1^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty$;

(γ) $\exists a \in \mathbb{R}$, $\exists b > 0$ such that $\omega(t) \geq a + b\log(1+t)$, $\forall t \geq 0$;

(δ) $\varphi_\omega(t) := \omega(e^t)$ is a convex function.

Sometimes, we also consider the stronger condition
(γ'): $\log(1+t) = o(\omega(t))$ as $t \to \infty$.
We define $\omega(z) = \omega(|z|)$, for $z \in \mathbb{C}^N$.

Weight functions (in the sense of BMT) 2/2

Definition

Given a non-quasianalytic weight function ω, we define the *Young conjugate* φ_ω^* of φ_ω as the function $\varphi_\omega^*\colon [0,\infty) \to [0,\infty)$ by

$$\varphi_\omega^*(s) := \sup_{t\geq 0}\{st - \varphi_\omega(t)\}, \quad s \geq 0.$$

The Young conjugate function φ_ω^* is convex and increasing, with $(\varphi_\omega^*)^* = \varphi_\omega$.

The space $\mathcal{S}_\omega(\mathbb{R}^N)$ (in the sense of Björck)

Definition

$\mathcal{S}_\omega(\mathbb{R}^N)$ is the Fréchet space of all functions $f \in L^1(\mathbb{R}^N)$ such that $f, \hat{f} \in C^\infty(\mathbb{R}^N)$ and $\forall \lambda > 0$ and $\forall \alpha \in \mathbb{N}_0^N$

$$\|e^{\lambda\omega}\partial^\alpha f\|_\infty < \infty \ \text{ and } \ \|e^{\lambda\omega}\partial^\alpha \hat{f}\|_\infty < \infty,$$

where $\hat{f}$ denotes the Fourier transform of f.

The elements of $\mathcal{S}_\omega(\mathbb{R}^N)$ are called *ω-ultradifferentiable rapidly decreasing functions of Beurling type*.

The spaces $\mathcal{E}_\omega(\mathbb{R}^N)$ and $\mathcal{D}_\omega(\mathbb{R}^N)$

Definition

$\mathcal{E}_\omega(\mathbb{R}^N)$ is the Fréchet space of all functions $f \in C^\infty(\mathbb{R}^N)$ such that

$$p_{K,\lambda}(f) := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^N} |\partial^\alpha f(x)| e^{-\lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)} < \infty$$

for all K compact subset of $\mathbb{R}^N$ and $\lambda > 0$.

The elements of $\mathcal{E}_\omega(\mathbb{R}^N)$ are called *ω-ultradifferentiable functions of Beurling type*.

In a natural way, one defines $\mathcal{D}_\omega(\mathbb{R}^N)$, whose elements are called *ω-test functions of Beurling type*.

The spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ 1/2

Definition

For $m \in \mathbb{N}, n \in \mathbb{Z}$, we define the Banach space $\mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$ as the set of all functions $f \in C^\infty(\mathbb{R}^N)$ satisfying the following condition:

$$r_{m,n}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| e^{-n\omega(x) - m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)} < \infty.$$

We define $\mathcal{O}_{M,\omega}(\mathbb{R}^N) := \underset{\overleftarrow{m}}{\mathit{proj}}\, \underset{\overrightarrow{n}}{\mathit{ind}}\, \mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$, which is a projective limit of (LB)-spaces.

The elements of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ are called *ω-slowly increasing functions of Beurling type*.

The spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ 2/2

Definition

We define $\mathcal{O}_{C,\omega}(\mathbb{R}^N) := \underset{\overrightarrow{n}}{\mathit{ind}}\, \underset{\overleftarrow{m}}{\mathit{proj}}\, \mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$, which is an (LF)-space.

The elements of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ are called *ω-very slowly increasing functions of Beurling type*.

Theorem

The inclusions

$$\mathcal{D}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{S}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{M,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$$

are well-defined, continuous and with dense range.

Description via the L^p-norms

Definition

For $m \in \mathbb{N}, n \in \mathbb{Z}$ and $p \in [1, \infty)$, we define the Banach space $\mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$ as the set of all functions $f \in C^\infty(\mathbb{R}^N)$ such that:

$$r^p_{m,n,p}(f) := \sum_{\alpha \in \mathbb{N}_0^N} \|e^{-n\omega}\partial^\alpha f\|_p^p e^{-mp\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)} < \infty.$$

Analogously ,we obtain the PLB-space $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ and the (LF)-space $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$.

Theorem

$\mathcal{O}_{M,\omega}(\mathbb{R}^N) = \mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N) = \mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ algebraically and topologically, for all $p \in [1, \infty)$.

$\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of all multipliers

Theorem

Consider $f \in C^\infty(\mathbb{R}^N)$. Then the following properties are equivalent:

1. $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$;
2. *For every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$, we have $fg \in \mathcal{S}_\omega(\mathbb{R}^N)$;*
3. *For every $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$, we have $fT \in \mathcal{S}'_\omega(\mathbb{R}^N)$.*

Moreover, if $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, then the linear operators $M_f \colon \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$ defined by $M_f(g) := fg$, for $g \in \mathcal{S}_\omega(\mathbb{R}^N)$, and $\mathcal{M}_f \colon \mathcal{S}'_\omega(\mathbb{R}^N) \to \mathcal{S}'_\omega(\mathbb{R}^N)$ defined by $\mathcal{M}_f(T) := fT$, for $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$, are continuous.

Topologies on $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$

Theorem

$f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ if, and only if, $\forall g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $\forall m \in \mathbb{N}$ we have

$$q_{m,g}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \|g\, \partial^\alpha f\|_\infty < \infty.$$

The set $\{q_{m,g}\}_{m \in \mathbb{N}, g \in \mathcal{S}_\omega(\mathbb{R}^N)}$ defines a complete Hausdorff lc-topology τ.
$\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ (via the map M) can be also endowed with either the topology τ_b or τ_s induced by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$ and $\mathcal{L}_s(\mathcal{S}_\omega(\mathbb{R}^N))$.
Due to a result of A. Debrouwere and L. Neyt[1], it follows that $t = \tau = \tau_b = \tau_s$, where t is the PLB topology.
$\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is a complete Montel PLB-space.

[1] *Weighted (PLB)-spaces of ultradifferentiable functions and multiplier spaces*, Monatsh Math **198**, 31–60 (2022)

$\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of all convolutors

Theorem

For $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$, consider the following conditions:

1. $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$;
2. *For every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, we have $T \star f \in \mathcal{S}_\omega(\mathbb{R}^N)$;*
3. *For every $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$, we have $T \star S \in \mathcal{S}'_\omega(\mathbb{R}^N)$.*

Then (1)⇒(2) and (2)⇔(3). If, in addition, ω satisfies the stronger condition (γ'), then (2)⇒(1).

Moreover, if $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$, then the linear operators $C_T \colon \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$ and $\mathcal{C}_T \colon \mathcal{S}'_\omega(\mathbb{R}^N) \to \mathcal{S}'_\omega(\mathbb{R}^N)$ defined by $C_T(f) := T \star f$, for $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $\mathcal{C}_T(S) := T \star S$, for $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$ are continuous.

The action of the Fourier Transform on the multiplier and convolutor spaces

Theorem (Albanese, M.)

Assume that ω satisfies the stronger condition (γ'). The Fourier transform $\mathcal{F}$ is a topological isomorphism from the space $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_b)$ onto the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Furthermore, for $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ and $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$, we have

$$\mathcal{F}(T \star S) = \mathcal{F}(T)\mathcal{F}(S),$$

and if $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$, we have

$$\mathcal{F}(fT) = (2\pi)^{-N} \hat{f} \star \mathcal{F}(T).$$

Multiplication topological algebras

Lemma

For all $k \in \mathcal{S}_\omega(\mathbb{R}^N)$ there exists $l \in \mathcal{S}_\omega(\mathbb{R}^N)$ such that $|k(x)| \leq l^2(x)$ for every $x \in \mathbb{R}^N$.

Theorem

The spaces $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \cdot)$, $(\mathcal{S}_\omega(\mathbb{R}^N), \cdot)$ are multiplication topological algebras.

Theorem

The space $(\mathcal{O}_{C,\omega}(\mathbb{R}^N), \cdot)$ is a multiplication algebra, but it is not a multiplication topological algebra.

Convolution topological algebras

Theorem

Assume that ω satisfies the stronger condition (γ'). Then $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \star)$, $(\mathcal{S}_\omega(\mathbb{R}^N), \star)$ are convolution topological algebras.

Theorem

The space $(\mathcal{O}'_{M,\omega}(\mathbb{R}^N), \star)$ is a convolution algebra, but it is not a convolution topological algebra.

Multiplier and convolutor spaces

E	M(E)	C(E)
$\mathcal{S}_\omega(\mathbb{R}^N)$	$\mathcal{O}_{M,\omega}(\mathbb{R}^N)$	$\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$
$\mathcal{S}'_\omega(\mathbb{R}^N)$	$\mathcal{O}_{M,\omega}(\mathbb{R}^N)$	$\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$
$\mathcal{O}_{M,\omega}(\mathbb{R}^N)$	$\mathcal{O}_{M,\omega}(\mathbb{R}^N)$	$\mathcal{O}'_{M,\omega}(\mathbb{R}^N)$
$\mathcal{O}'_{M,\omega}(\mathbb{R}^N)$	$\mathcal{O}_{M,\omega}(\mathbb{R}^N)$	$\mathcal{O}'_{M,\omega}(\mathbb{R}^N)$
$\mathcal{O}_{C,\omega}(\mathbb{R}^N)$	$\mathcal{O}_{C,\omega}(\mathbb{R}^N)$	$\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$
$\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$	$\mathcal{O}_{C,\omega}(\mathbb{R}^N)$	$\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$

Topologies on $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$

We consider the Fréchet space $\mathcal{O}_{n,\omega,p}(\mathbb{R}^N) := \operatorname{proj}_{\overleftarrow{m}} \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$, for $n \in \mathbb{N}$ and $1 < p < \infty$. The family $\{\mathcal{O}_{n,\omega,p}(\mathbb{R}^N)\}_{n\in\mathbb{N}}$ is a reduced inductive spectrum of reflexive Fréchet spaces.

Hence, we can define on the space $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ the projective topology τ_{pr} via its projective spectrum $\{\mathcal{O}'_{n,\omega,p}(\mathbb{R}^N)\}_{n\in\mathbb{N}}$.

Proposition

The inclusion $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_{pr}) \hookrightarrow (\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_b)$ is continuous.

Theorem

Assume that ω satisfies the stronger condition (γ'). Then $\tau_\beta = \tau_{pr} = \tau_b$ on $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ (τ_β is the strong topology).

Proof 1/3

For all $n \in \mathbb{N}$ we set

$$U_n := \{f \in \mathcal{S}_\omega(\mathbb{R}^N) \colon q_{n,n}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \|e^{n\omega}\partial^\alpha f\|_\infty e^{-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)} \leq 1\}.$$

The family $\{U_n\}_{n\in\mathbb{N}}$ is a basis of closed absolutely convex 0-neighborhoods of $\mathcal{S}_\omega(\mathbb{R}^N)$.

It suffices to show that for all $n \in \mathbb{N}$ and every bounded, closed absolutely convex subset B of $\mathcal{S}_\omega(\mathbb{R}^N)$, the set

$$M(B, U_n) := \{T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \colon T \star f \in U_n, \forall f \in B\}$$

contains $\overset{\circ}{C}$ (the polar taken in $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$), for some bounded subset C of $\mathcal{O}_{n',\omega,2}(\mathbb{R}^N)$ and some $n' \geq n$.

Proof 2/3

For a fixed $n \in \mathbb{N}$, we observe that if $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$, then $T \in ((\mathcal{O}^m_{n',\omega,2}(\mathbb{R}^N))', r'_{m,n',2})$, for some $m \in \mathbb{N}$ which can be always supposed greater or equal than $n' = [Kn] + 1 \geq n$.

Due to a structure theorem, there exists $\{f_\alpha\}_{\alpha \in \mathbb{N}_0^N}$ such that $T = \sum_{\alpha \in \mathbb{N}_0^N} \partial^\alpha f_\alpha$ and $\sum_{\alpha \in \mathbb{N}_0^N} \|e^{n'\omega} f_\alpha\|_2^2 e^{2m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)} < \infty$.

Fixed a bounded closed absolutely convex subset B of $\mathcal{S}_\omega(\mathbb{R}^N)$, we obtain for every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ that

$$\sup_{f \in B} q_{n,n}(T \star f) \leq c \left(\sup_{f \in B} \sigma_{2m,n',2}(f) \right) r'_{m,n',2}(T),$$

where $\sigma^2_{2m,n',2}(f) = \sum_{\alpha \in \mathbb{N}_0^N} \|e^{n'\omega} \partial^\alpha f\|_2^2 e^{-4m\varphi^*_\omega\left(\frac{|\alpha|}{2m}\right)}$ and $c > 0$.

Proof 3/3

For each $m \geq n'$, set $\lambda_m := c \sup_{f \in B} \sigma_{2m,n',2}(f) < \infty$ and $V_{m,n'} := \{f \in \mathcal{O}_{n',\omega,2}(\mathbb{R}^N) \colon r_{m,n',2}(f) \leq 1\}$. Then $C := \cap_{m \geq n'} \lambda_m V_{m,n'}$ is a bounded closed absolutely convex subset of $\mathcal{O}_{n',\omega,2}(\mathbb{R}^N)$.

Now, if $T \in \Gamma(\cup_{m \geq n'} \lambda_m^{-1} \overset{\circ}{V}_{m,n'})$, then $T = \sum_{m \geq n'} \alpha_m \lambda_m^{-1} T_m$, with $T_m \in \overset{\circ}{V}_{m,n'}$ for all $m \geq n'$ and $\sum_{m \geq n'} |\alpha_m| \leq 1$. This implies $\sup_{f \in B} q_{n,n}(T \star f) \leq 1$, i.e., $T \in M(B, U_n)$.

Since $M(B, U_n)$ is $\sigma := \sigma(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \mathcal{O}_{C,\omega}(\mathbb{R}^N))$-closed, it follows that $\overset{\circ}{C} \subseteq M(B, U_n)$, being $\overset{\circ}{C} = \overline{\Gamma(\cup_{m \geq n'} \lambda_m^{-1} \overset{\circ}{V}_{m,n'})}^{\sigma}$. $\square$

Definitions

An (LF)-space $E = \operatorname{ind}_n E_n$ is called

1. *sequentially retractive*, if every convergent sequence in E is contained in some step E_n and converges there;
2. *quasi-regular*, if for every bounded subset B of E there exists $n \in \mathbb{N}$ and a bounded subset C of E_n such that $B \subseteq \overline{C}$, where the closure is taken in E;
3. *regular*, if every bounded subset in E is contained and bounded in E_n, for some $n \in \mathbb{N}$.

$\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a complete (LF)-space

Theorem

Assume that ω satisfies the stronger condition (γ'). Then $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a sequentially rectractive Montel (LF)-space. In particular, $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a complete (LF)-space.

Lemma

The Fréchet spaces $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$ and $\mathcal{E}_\omega(\mathbb{R}^N)$ induce the same topology on the bounded subsets of $\mathcal{O}_{n,\omega}(\mathbb{R}^N)$.

Proof 1/3

Firstly, we show that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is quasi-regular.

If B is a bounded closed absolutely convex subset of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, then its polar $\overset{\circ}{B}$ is a 0-neighborhood of $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_\beta)$. But $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_\beta) = \operatorname{proj}_{\overset{\leftarrow}{n}} \mathcal{O}'_{n,\omega,p}(\mathbb{R}^N)$, for any $1 < p < \infty$. Hence, there exists a bounded absolutely convex subset C of $\mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$ for some $n \in \mathbb{N}$ such that $\overset{\circ}{C} \subseteq \overset{\circ}{B}$, thereby implying by the bipolar theorem that $B = \overset{\circ\circ}{B} \subseteq \overset{\circ\circ}{C} = \overline{C}$.

Proof 2/3

Now, we show that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is regular.

Let B be a bounded subset of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and let C be a bounded subset of $\mathcal{O}_{n,\omega}(\mathbb{R}^N)$ for some $n \in \mathbb{N}$ such that $B \subseteq \overline{C}$. Let $\overline{C}^{n+1}$ be the closure of C in $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$. The Fréchet spaces $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$ and $\mathcal{E}_\omega(\mathbb{R}^N)$ induce the same topology on C. Therefore, the closure of C in $\mathcal{E}_\omega(\mathbb{R}^N)$ coincides with $\overline{C}^{n+1}$. As $\mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ continuously, it follows that $\overline{C}^{n+1}$ is also a closed subset of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and hence, $B \subseteq \overline{C} \subseteq \overline{C}^{n+1}$. This implies that B is contained and bounded in $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$.

Proof 3/3

Finally, we show that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is sequentially retractive.

Let $\{f_j\}_j$ be a null sequence in $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$. Since $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a regular (LF)-space, there exists $n \in \mathbb{N}$ such that $\{f_j\}_j$ is contained and bounded in $\mathcal{O}_{n,\omega}(\mathbb{R}^N)$. We observe that $\{f_j\}_j$ is a null sequence also in $\mathcal{E}_\omega(\mathbb{R}^N)$. Therefore, $\{f_j\}_j$ is a null sequence of $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$. This shows that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a sequentially retractive (LF)-space.

The completeness follows by a result of Wengenroth.

The fact that the bounded subsets of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ are relatively compact follows by arguing as done for the null sequences. $\square$

Main references

1. A.A. Albanese, C. Mele, *Multipliers on $\mathcal{S}_\omega(\mathbb{R}^N)$*, J. Pseudo-Differ. Oper. Appl. **12** (2021), Article 35.
2. A.A. Albanese, C. Mele, *Convolutors on $\mathcal{S}_\omega(\mathbb{R}^N)$*, RACSAM **115** (2021), Article 157.
3. A.A. Albanese, C. Mele, *Multiplication and convolution topological algebras in spaces of ω-ultradifferentiable function of Beurling type*, in: Recent Advances in Mathematical Analysis, Trends in Mathematics (to appear).
4. A.A Albanese, C. Mele, *On the space $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and its dual*, preprint 2022.
5. A. Debrouwere, J. Vindas, *On weighted inductive limits of spaces of ultradifferentiable functions and their duals*, Math. Nachr. **292** (2019), 573–602.
6. J. Wengenroth, *Acyclic inductive spectra of Fréchet spaces*, Studia Math. **120** (1996), 247–258.