Optimal flat functions in Carleman-Roumieu ultraholomorphic classes in sectors

Ignacio Miguel (University of Valladolid, Spain)

Joint work with J. Jiménez-Garrido (Univ. Cantabria),
J. Sanz (Univ. Valladolid),
G. Schindl (Univ. Vienna)

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Sectors and log-convex sequences

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$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Let $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers, with $M_0 = 1$.

$\mathbb{M}$ is said to be logarithmically convex or (lc) if $M_n^2 \le M_{n-1}M_{n+1},\ n \ge 1$; equivalently, the sequence of quotients of $\mathbb{M}$, $\boldsymbol{m} = (m_n := \frac{M_{n+1}}{M_n})_{n\in\mathbb{N}_0}$, is nondecreasing.

Weight sequences

We always assume that $\mathbb{M}$ is (lc) and $\lim_{n\to\infty} m_n = \infty$: we say $\mathbb{M}$ is a weight sequence.

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Examples:

- $\mathbb{M} = (\prod_{k=0}^{n} \log^{\beta}(e+k))_{n\in\mathbb{N}_0}$, $\beta > 0$, $m_n = \log^{\beta}(e+n+1)$.
- $\mathbb{M}_\alpha = (n!^{\alpha})_{n\in\mathbb{N}_0}$, Gevrey sequence of order $\alpha > 0$, $m_n = (n+1)^{\alpha}$.
- $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$, $m_n = (n+1)^{\alpha} \log^{\beta}(e+n+1)$.
- For $q > 1$, $\mathbb{M} = (q^{n^2})_{n\in\mathbb{N}_0}$, $q-$Gevrey sequence, $m_n = q^{2n+1}$.

Asymptotics

$f : S \to \mathbb{C}$ (holomorphic in a sector S) admits the series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ as its $\mathbb{M}$-uniform asymptotic expansion at 0 if there exist $C, A > 0$ such that for every $z \in S$ and every $n \in \mathbb{N}_0$, we have

$$\Big|f(z) - \sum_{k=0}^{n-1} a_k z^k\Big| \leq C A^n M_n |z|^n. \qquad [f \in \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)]$$

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The norm

$$\|f\|_{\mathbb{M},A,\widetilde{u}} := \sup_{z\in S, n\in\mathbb{N}_0} \frac{|f(z) - \sum_{k=0}^{n-1} a_k z^k|}{A^n M_n |z|^n}$$

makes it a Banach space ($\frac{1}{A}$ may be called the type).

$\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S) := \bigcup_{A>0} \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)$ is an (LB) space.

The Borel map

$\mathbb{C}[[z]]$ formal complex power series.

$$\mathbb{C}[[z]]_{\{\mathbb{M}\},A} = \left\{ \widehat{f} = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]] : \left| \widehat{f} \right|_{\mathbb{M},A} := \sup_{p\in\mathbb{N}_0} \frac{|a_p|}{A^p M_p} < \infty \right\}.$$

$(\mathbb{C}[[z]]_{\{\mathbb{M}\},A}, |\cdot|_{\mathbb{M},A})$ is a Banach space.

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We consider the asymptotic Borel map (continuous homomorphism of algebras)

$$\begin{aligned} \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S) \quad &\longrightarrow \quad \mathbb{C}[[z]]_{\{\mathbb{M}\}} \\ f \quad &\mapsto \hat{f} = \sum_{n=0}^{\infty} a_n z^n. \end{aligned}$$

It may also be considered from $\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)$ into $\mathbb{C}[[z]]_{\{\mathbb{M}\},A}$.

Surjectivity intervals and its non-triviality

$$\widetilde{S}^{u}_{\{\mathbb{M}\}} := \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^{u}_{\{\mathbb{M}\}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\{\mathbb{M}\}} \text{ is surjective}\}.$$

$\widetilde{S}^{u}_{\{\mathbb{M}\}}$ is either empty, or interval having 0 as left-endpoint.

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$\mathbb{M}$ is strongly non-quasianalytic (snq) if there exists $B > 0$ such that

$$\sum_{k \geq n} \frac{M_k}{(k+1)M_{k+1}} \leq B \frac{M_n}{M_{n+1}}, \quad n \in \mathbb{N}_0.$$

H.-J. Petzsche, On E. Borel's theorem, Math. Ann. 282 (1988), no. 2, 299–313.

V. Thilliez (2003)

If $\mathbb{M}$ does not satisfy (snq), $\widetilde{S}^u_{\{\mathbb{M}\}} = \emptyset$.

Thilliez's index

V. Thilliez (2003) introduces a growth index $\gamma(\mathbb{M})$. Now we know:
A sequence $(c_p)_{p\in\mathbb{N}_0}$ of positive real numbers, is almost increasing if there exists $a>0$ such that for every $p\in\mathbb{N}_0$ we have that $c_p\le ac_q$ for every $q\ge p$.
We have proved that

$$\begin{aligned}\gamma(\mathbb{M}) &= \sup\{\gamma>0 : (m_p/(p+1)^{\gamma})_{p\in\mathbb{N}_0} \text{ is almost increasing}\}\\ &=: \text{lower Matuszewska index of } \boldsymbol{m}.\end{aligned}$$

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Moreover, $\gamma(\mathbb{M})>0$ if and only if $\mathbb{M}$ is (snq).

Optimal flat functions and associated functions

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For any sequence $\mathbb{M}$ we can consider the map $h_{\mathbb{M}} : [0, \infty) \to \mathbb{R}$, defined by

$$h_{\mathbb{M}}(t) := \inf_{k \in \mathbb{N}_0} M_k t^k, \quad t > 0; \quad h_{\mathbb{M}}(0) = 0.$$

Let $f \in \mathcal{H}(S)$, the following are equivalent:

1. $f \in \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S)$ and it is flat.
2. $|f(z)| \leq C h_{\mathbb{M}}(K|z|)$, for some $C, K \in \mathbb{R}$, and for all $z \in S$.

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Definition

Let $\mathbb{M}$ a weight sequence, S an unbounded sector bisected by the positive real line $(0,+\infty)$. A function $G \in \mathcal{H}(S)$ is called an *optimal flat function*, if

(i) $\exists K_1, K_2 > 0 : \quad K_1 h_{\mathbb{M}}(K_2 x) \leq G(x)$ for all $x > 0$,

(ii) $\exists K_3, K_4 > 0 : \quad |G(z)| \leq K_3 h_{\mathbb{M}}(K_4|z|)$ for all $z \in S$.

Surjectivity intervals for strongly regular sequences

$\mathbb{M}$ is strongly regular if it is (lc), (snq) and has moderate growth (mg): there exists $A > 0$ such that $M_{n+p} \leq A^{n+p} M_n M_p$, $n, p \in \mathbb{N}_0$.

Example: $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.

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Theorem (V. Thilliez, 2003)

Let $\mathbb{M}$ be a strongly regular sequence. Then, $\gamma(\mathbb{M}) \in (0,\infty)$. Moreover, each of the following statements implies the next one:

(i) $0 < \gamma < \gamma(\mathbb{M})$,

(ii) *the space $\widetilde{\mathcal{A}}^{u}_{\{\mathbb{M}\}}(S_\gamma)$ contains optimal flat functions,*

(iii) *there exists $c \ge 1$, depending on $\mathbb{M}$ and γ, such that for every $A > 0$ there exists a right inverse for $\widetilde{\mathcal{B}}$, $U_{\mathbb{M},A,\gamma} : \mathbb{C}[[z]]_{\{\mathbb{M}\},A} \to \widetilde{\mathcal{A}}^{u}_{\{\mathbb{M}\},cA}(S_\gamma)$,*

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It is not known whether $\gamma(\mathbb{M})$ belongs or not to the surjectivity intervals.

Results for regular sequences in the sense of E. M. Dyn'kin

E. M. Dyn'kin, Pseudoanalytic extension of smooth functions. The uniform scale, Amer. Math. Soc. Transl. (2) 115 (1980), 33–58.

$\mathbb{M}$ is derivation closed (dc) if there exists a constant $A > 0$ such that

$$M_{n+1} \leq A^{n+1} M_n, \quad n \in \mathbb{N}_0.$$

$\widehat{\mathbb{M}} := (n! M_n)_{n\in\mathbb{N}_0}$ is regular if $\mathbb{M}$ is a weight sequence and satisfies (dc).
If $\mathbb{M}$ is strongly regular, the corresponding $\widehat{\mathbb{M}}$ is regular.

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If $\mathbb{M}$ is strongly regular, the corresponding $\widehat{\mathbb{M}}$ is regular.

No proof of surjectivity had been given for regular $\widehat{\mathbb{M}}$, except for the q-Gevrey sequences $\mathbb{M} = (q^{n^2})_{n\in\mathbb{N}_0}$, $q > 1$, see
C. Zhang, Développements asymptotiques q-Gevrey et séries Gq-sommables, Ann. Inst. Fourier 49 (1999), 227–261.

Connection with the Stieltjes moment problem

A. Debrouwere, J. Jiménez-Garrido, J. Sanz, Injectivity and surjectivity of the Stieltjes moment mapping in Gelfand-Shilov spaces, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 113 (2019), 3341–3358, DOI: 10.1007/s13398-019-00693-6.

By a suitable application of the Fourier transform, there exists a close connection between this problem and the surjectivity or injectivity of the asymptotic Borel map in ultraholomorphic classes in a half-plane, and so their results in JMAA19 could be transferred.

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A. Debrouwere, Solution to the Stieltjes moment problem in Gelfand-Shilov spaces, Studia Math. 254 (2020), 295-323, DOI: 10.4064/sm190627-8-10.

He has got a characterization of the surjectivity of the Stieltjes moment mapping for regular sequences by using only functional-analytic methods.

Theorem (A. Debrouwere, 2020)

Let $\widehat{\mathbb{M}}$ be regular. The following are equivalent:

(i) $\widetilde{\mathcal{B}}\colon \mathcal{A}^u_{\{\mathbb{M}\}}(S_1) \to \mathbb{C}[[z]]_{\{\mathbb{M}\}}$ *is surjective.*

(ii) $\gamma(\mathbb{M}) > 1$.

Surjectivity intervals for regular sequences

By using Balser's moment summability methods, with associated Laplace and Borel transforms, they proved

Theorem (J. Jiménez-Garrido, J. Sanz, G. Schindl, 2021)

Let $\widehat{\mathbb{M}}$ *be a regular sequence. Then,*

$$(0,\gamma(\mathbb{M})) \subseteq \widetilde{S}^{u}_{\{\mathbb{M}\}} \subseteq (0,\gamma(\mathbb{M})].$$

In general it is not known whether $\gamma(\mathbb{M})$ belongs or not to the surjectivity intervals.

Conjecture: $\widetilde{S}^{u}_{\{\mathbb{M}\}} = (0,\gamma(\mathbb{M}))$ in general.

Open problem

Aim: Obtain a constructive proof for the surjectivity of the Borel map. We hope to obtain a similar result to Thilliez's for regular sequences.

Construction of optimal flat functions: Preliminaries

Associated functions

Given a weight sequence $\mathbb{M}$, we consider this two associated functions:

- The associated weight function $\omega_{\mathbb{M}}(t) := \sup_{p\in\mathbb{N}_0} \log\left(\frac{t^p}{M_p}\right)$ for all $t \geq 0$. Note that there is a relation between $\omega_{\mathbb{M}}$ and $h_{\mathbb{M}}$, that is, $h_{\mathbb{M}}(t) = \exp(-\omega_{\mathbb{M}}(1/t))$ for all $t > 0$.
- The counting function associated with $\mathbb{M}$, that is $\nu_{\boldsymbol{m}}(t) = \#\{n \in \mathbb{N}_0 : m_n \leq t\}$ for all $t \geq 0$.

Construction of optimal flat functions: Preliminaries

Definition

Let $\mathbb{M}$ be a weight sequence with $\gamma(\mathbb{M}) > 0$. We consider the harmonic extension of $\sigma = \omega_{\mathbb{M}}$ or $\sigma = \nu_{\boldsymbol{m}}$ to the open upper half plane, that is, for all $x \in \mathbb{R}$ and $y \geq 0$:

$$P_\sigma(x+iy) = \begin{cases} \sigma(x) & y = 0 \\ \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\sigma(|t|)}{(t-x)^2+y^2}\,dt & y > 0. \end{cases}$$

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Note that $\omega_{\mathbb{M}}(|x+iy|) \leq P_{\omega_{\mathbb{M}}}(x+iy)$, for all $x \in \mathbb{R}$ and $y \geq 0$.

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The Langenbruch condition (1994)

Let $\mathbb{M}$ be a weight sequence with $\gamma(\mathbb{M}) > 0$. There exist a constant $C > 0$ such that for all $y \geq 0$ we have that:

$$P_{\omega_{\mathbb{M}}}(iy) \leq \omega_{\mathbb{M}}(Cy) + C. \qquad (\mathcal{L})$$

Construction of optimal flat functions I

Proposition (Case S_1)

Let $\mathbb{M}$ be a weight sequence with $\gamma(\mathbb{M}) > 0$. If the condition $(\mathcal{L})$ holds, then the function $G(i/z) = \exp(-P_{\omega_{\mathbb{M}}}(i/z) + iQ_{\omega_{\mathbb{M}}}(i/z))$ is an optimal flat function in S_1, where $Q_{\omega_{\mathbb{M}}}$ is the harmonic conjugate of $P_{\omega_{\mathbb{M}}}$ in the upper half plane.

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- For all $z \in S_1$, we have that

$$|G(z)| = \exp(-P_{\omega_{\mathbb{M}}}(i/z)) \leq \exp(-\omega_{\mathbb{M}}(1/|z|)) = h_{\mathbb{M}}(|z|).$$

- Let consider $x > 0$, then by Langenbruch condition, we deduce that:

$$|G(x)| = \exp(-P_{\omega_{\mathbb{M}}}(i/x)) \geq \exp(-\omega_{\mathbb{M}}(C/x) - C) = \exp(-C)h_{\mathbb{M}}(x/C).$$

The Langenbruch condition and the index $\gamma(\mathbb{M})$

Proposition (D. N. Nenning, A. Rainer and G. Schindl 2022)

Let $\widehat{\mathbb{M}}$ be a regular sequence, with $\gamma(\mathbb{M}) > 0$. The following are equivalent:

- $\gamma(\mathbb{M}) > 1$.
- *There exist a constant $C > 0$ such that for all $y \geq 0$ we have the Langenbruch condition (1994):*

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Let $\mathbb{M}$ be a weight sequence, with $\gamma(\mathbb{M}) > 0$. The following are equivalent:

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Sketch of the proof: Auxiliary results

Definition

Let $\mathbb{M}$ be a weight sequence, we consider the function $\sigma = \omega_{\mathbb{M}}$ or $\sigma = \nu_{\boldsymbol{m}}$. We define the function $\kappa_\sigma(y)$ by:

$$\kappa_\sigma(y) = \int_1^\infty \frac{\sigma(ys)}{s^2} ds \qquad y > 0.$$

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Theorem

Let $\mathbb{M}$ be a weight sequence with $\gamma(\mathbb{M}) > 0$, we consider the function $\sigma = \omega_{\mathbb{M}}$ or $\sigma = \nu_{\boldsymbol{m}}$. Then, we have these estimations:

$$\frac{1}{\pi}\kappa_\sigma(y) \le P_\sigma(iy) \le \kappa_\sigma(y) \qquad y > 0.$$

Sketch of the proof: $(2) \Rightarrow (1)$

First, if we use the relation between the functions $\omega_{\mathbb{M}}$ and $\nu_{\boldsymbol{m}}$, we deduce this estimations for all $r \geq 0$ and $B \geq 0$:

$$\omega_{\mathbb{M}}(e^B r) = \int_0^{e^B r} \frac{\nu_{\boldsymbol{m}}(u)}{u} du = \omega_{\mathbb{M}}(r) + \int_r^{e^B r} \frac{\nu_{\boldsymbol{m}}(u)}{u} du \geq \omega_{\mathbb{M}}(r) + B\nu_{\boldsymbol{m}}(r).$$

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$$\omega_{\mathbb{M}}(e^B r) = \int_0^{e^B r} \frac{\nu_{\boldsymbol{m}}(u)}{u} du = \omega_{\mathbb{M}}(r) + \int_r^{e^B r} \frac{\nu_{\boldsymbol{m}}(u)}{u} du \geq \omega_{\mathbb{M}}(r) + B\nu_{\boldsymbol{m}}(r).$$

Then, for all $y \geq 0$ we observe that:

$$\omega_{\mathbb{M}}(y) + \kappa_{\nu_{\boldsymbol{m}}}(y) \leq P_{\omega_{\mathbb{M}}+\pi\nu_{\boldsymbol{m}}}(iy) \leq P_{\omega_{\mathbb{M}}(e^{\pi}\cdot)} = P_{\omega_{\mathbb{M}}}(ie^{\pi}y) \leq \omega_{\mathbb{M}}(Ce^{\pi}y) + C.$$

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Finally, thanks to the fact that $\nu_{\boldsymbol{m}}$ satisfies the condition $\nu_{\boldsymbol{m}}(2t) = O(\nu_{\boldsymbol{m}}(t))$ for t tends to ∞, we have that:

$$\begin{aligned}\kappa_{\nu_{\boldsymbol{m}}}(y) &\leq \omega_{\mathbb{M}}(Ce^{\pi}y) - \omega_{\mathbb{M}}(y) + C = \int_y^{Ce^{\pi}y} \frac{\nu_{\boldsymbol{m}}(u)}{u} du + C \\ &\leq \nu_{\boldsymbol{m}}(Ce^{\pi}y) + \ln(Ce^{\pi}) + C \leq D\nu_{\boldsymbol{m}}(y) + D \qquad y \geq 0,\ D > 0.\end{aligned}$$

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We conclude that $\gamma(\mathbb{M}) = \gamma(\nu_{\boldsymbol{m}}) > 1$.

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$$\kappa_{\omega_{\mathbb{M}}}(y) = \omega_{\mathbb{M}}(y) + \kappa_{\nu_{\boldsymbol{m}}}(y) \qquad y \geq 0.$$

Moreover, setting $B = 1$, we obtained in the previous case this estimations:

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The condition $\gamma(\mathbb{M}) > 1$ is equivalent to the fact that for all $y \geq 0$ there exist a constant $C \in \mathbb{N}$ such that $\kappa_{\nu_{\boldsymbol{m}}}(y) \leq C\nu_{\boldsymbol{m}}(y)$.Then, from the auxiliary result and the above identities we deduce that:

$$\begin{aligned} P_{\omega_{\mathbb{M}}}(iy) \leq \kappa_{\omega_{\mathbb{M}}}(y) &= \omega_{\mathbb{M}}(y) + \kappa_{\nu_{\boldsymbol{m}}}(y) \leq \omega_{\mathbb{M}}(y) + C\nu_{\boldsymbol{m}}(y) \\ &= \omega_{\mathbb{M}}(y) + \nu_{\boldsymbol{m}}(y) + (C-1)\nu_{\boldsymbol{m}}(y) \\ &\leq \omega_{\mathbb{M}}(ey) + \nu_{\boldsymbol{m}}(ey) + (C-2)\nu_{\boldsymbol{m}}(y) \\ &\dots \\ &\leq \omega_{\mathbb{M}}(e^{C}y) \qquad\qquad \forall y \geq 0. \end{aligned}$$

Construction of optimal flat functions II

Proposition

Let $\mathbb{M}$ be a weight sequence with $0 < \gamma(\mathbb{M})$. Then, for any $0 < \gamma < \gamma(\mathbb{M})$ there exist an optimal flat function in S_γ.

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Let $\mathbb{M}$ be a weight sequence with $0 < \gamma(\mathbb{M})$. Then, for any $0 < \gamma < \gamma(\mathbb{M})$ there exist an optimal flat function in S_γ.

- Fix $s > 0$ such that $\gamma < 1/s < \gamma(\mathbb{M})$. Then, we note that $\gamma(\mathbb{M}^s) = s\gamma(\mathbb{M}) > 1$.
- We apply the last result to the sequence $\mathbb{M}^s$. There exist an optimal flat function $G(z)$ in S_1. It is important to regard that the bounds will be in terms of $h_{\mathbb{M}^s}$, instead of $h_{\mathbb{M}}$.
- Now, we consider the function $F(z) = (G(z^s))^{1/s}$ for all $z \in S_\gamma$. From the definition of F, G and the relation between de functions $\omega_{\mathbb{M}^s}$ and $\omega_{\mathbb{M}}$, that is:

$$\omega_{\mathbb{M}}(t^{1/s}) = \frac{1}{s}\omega_{\mathbb{M}^s}(t) \qquad t \geq 0,$$

we prove that the function F is an optimal flat function in S_γ.

Surjectivity intervals for regular sequences

Theorem

Let $\widehat{\mathbb{M}}$ be a regular sequence with $\gamma(\mathbb{M}) \in (0,\infty]$. Moreover, each of the following statements implies the next one:

(i) $0 < \gamma < \gamma(\mathbb{M})$,

(ii) *the space $\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_\gamma)$ contains optimal flat functions,*

(iii) *there exists $c \geq 1$, depending on $\mathbb{M}$ and γ, such that for every $A > 0$ there exists a right inverse for $\widetilde{\mathcal{B}}$, $U_{\mathbb{M},A,\gamma} : \mathbb{C}[[z]]_{\{\mathbb{M}\},A} \to \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},cA}(S_\gamma)$,*

(iv) $\widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\{\mathbb{M}\}}$ *is surjective,*

(v) $0 < \gamma \leq \gamma(\mathbb{M})$.

We need (dc) condition for $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v)$.

THANK YOU VERY MUCH FOR YOUR ATTENTION!