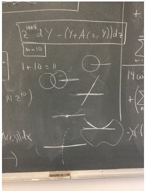

Nilpotent singularities of holomorphic foliations

New results on two problems

Workshop on Functional and Complex Analysis
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Contents

Main notions

- A germ of holomorphic foliation of codimension one on $(\mathbb{C}^n, \mathbf{0})$ is defined by a holomorphic 1-form ω, verifying the Frobenius integrability condition $\omega \wedge d\omega = 0$. We shall assume that $\omega = \sum_{i=1}^n a_i(\mathbf{x}) dx_i$, where the coefficients $a_i(\mathbf{x})$ have no common factors

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- If an invertible formal transformation $\hat{\Phi} = (\Phi_1, \Phi_2) \in \mathbb{C}[[x,y]]^2$ exists such that $\Phi^*\omega_1 \wedge \omega_2 = 0$, the foliations are said to be formally equivalent.

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Example

Consider Euler's foliation

$$\omega = x^2 dy + (y - x)dx.$$

The formal change of variables $\Phi^*(x, y) = (x, y + \sum_{n=0}^{\infty}(-1)^n n! x^{n+1})$ verifies that $\Phi^*\omega = x^2 dy + y dx$. These 1-forms are not analytically equivalent.

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- Analytically, f is a separatrix of ω if and only if there exists a 2-form η such that $\omega \wedge df = f\eta$. Equivalently if there are germs of functions g, k and a 1-form η such that

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  and moreover $k \not\equiv 0$ on $\mathcal{C}$.
- Previous expressions allow to extend this notion to the formal case.
- A two-dimensional foliation always has at least one analytic separatrix (Camacho-Sad Theorem). It may have finitely or infinitely many of them.

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$$\omega \wedge d(y^2 + x^3) = -6A(x, y)(y^2 + x^3)dx \wedge dy.$$

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every curve $y^2 + cx^3$ is a separatrix for $2xdy - 3ydx$.
- The situation where infinitely many separatrices appear, in dimension 2, is called dicritical.

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- Consider the foliation defined by a vector field $\mathcal{X} = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$, and let M be the matrix of its linear part, i.e.

$$M = \begin{pmatrix} \frac{\partial a}{\partial x}(0,0) & \frac{\partial a}{\partial y}(0,0) \\ \frac{\partial b}{\partial x}(0,0) & \frac{\partial b}{\partial y}(0,0) \end{pmatrix}.$$

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- When $\lambda_1 = 0$, the singularity is called a saddle-node. In the other case, it will be called hyperbolic.

An example

As an example, let us show how the reduction of singularities works in a simple case: $\omega = d(y^2 - x^3)$.

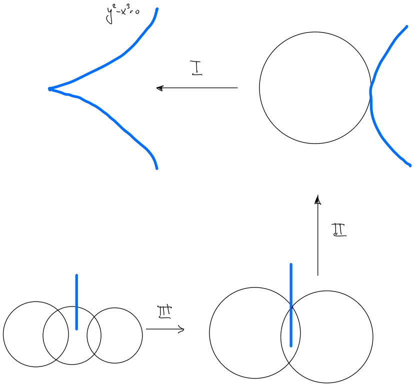

Figure: Reduction of the singularities of a cusp

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- In this talk we will introduce two problems. The first one concerns the analytic classification in the case $2p = n$. The second one concerns a family of foliations of generalized saddle-node type.

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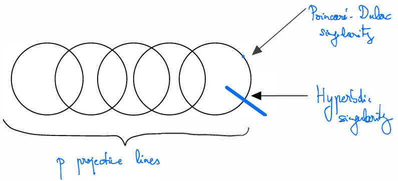

Figure: Reduction of the singularities of a generalized Poincaré Dulac singularity. First steps

Analytic Classification Theorem

Theorem (P. Fernández, –)

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two germs of generalized Poincaré-Dulac holomorphic foliations, formally equivalent. Assume that H_i is the holonomy group of the pth component of the exceptional divisor obtained when doing the reduction of singularities for $\mathcal{F}_i$ $(i = 1, 2)$. If H_1, H_2 are analytically conjugated, the foliations are also analytically conjugated.

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- This group has two generators, h_1, h_2. They are the local generators of the holonomy of a Poincaré-Dulac singularity and of a hyperbolic singularity, respectively.
- Due to the normal form of the Poincaré-Dulac singularity, it can be proved that there exists an **analytic vector field** $\mathcal{Y}$ such that

$$h_1 = \mu \exp(\mathcal{Y}), \qquad h_2 = \frac{\lambda}{\mu} \exp(-\mathcal{Y}),$$

where $\mu^m = 1$, $\lambda^p = 1$.

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- As a consequence, the projective holonomy group turns out to be:
  - Abelian, if p divides m.
  - Non solvable, if p does not divide m.

Projective foliations with only one singular point

- A global foliation in the projective plane $\mathbb{P}^2_{\mathbb{C}}$ is defined by a 1-form

$$A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz,$$

where A, B and C are homogeneous polynomials of degree d, satisfying the Euler relation $xA + yB + zC = 0$. Alternatively, vector fields can be used to define them.

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- Such a family of foliations, assuming non-zero linear part, is given by the following vector field [**A**]:
$$\mathcal{X} = \alpha y^d \frac{\partial}{\partial x} + \left(\beta x^{\frac{d-1}{2}} y z^{\frac{d-1}{2}} - \beta^2 z^d\right) \frac{\partial}{\partial y} + \left(x^{d-1} y - \beta x^{\frac{d-1}{2}} z^{\frac{d+1}{2}}\right) \frac{\partial}{\partial z}, \tag{1}$$
for odd d.

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- Around the only singular point, in local coordinates, this foliation may be written as

$$(y-\beta z^{\frac{d+1}{2}}-\alpha y^d z)dy+(\alpha y^{d+1}-\beta y z^{\frac{d-1}{2}}+\beta^2 z^d)dz.$$

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- Typically, it is very hard to know, in precise examples, if the formal separatrices are, in fact, divergent or not, even while divergence is generic.
- To study the local behaviour of the separatrices, several authors have introduced different invariants associated to them: indices.

Camacho-Sad indices

- If $\mathcal{C}$ is a separatrix of the foliation defined by the 1-form ω, and we write $g\omega = kdf + f\eta$, it can be defined the Camacho-Sad index of $\mathcal{C}$ at P as

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- In the particular (and important) case where $\mathcal{C}$ is smooth, $y = 0$ in appropriate coordinates, and $\omega = yp(y,z)dz + q(y,z)dy$, previous formula reduces to

$$CS_P(\mathcal{F}, \mathcal{C}) = -\mathrm{Res}\left(\frac{p(0,z)}{q(0,z)}; z = 0\right).$$

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- If $\mathcal{C}$ is a separatrix of the foliation defined by the 1-form ω, and we write $g\omega = kdf + f\eta$, it can be defined the Camacho-Sad index of $\mathcal{C}$ at P as
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- In the particular (and important) case where $\mathcal{C}$ is smooth, $y = 0$ in appropriate coordinates, and $\omega = yp(y,z)dz + q(y,z)dy$, previous formula reduces to
$$CS_P(\mathcal{F}, \mathcal{C}) = -\mathrm{Res}\left(\frac{p(0,z)}{q(0,z)}; z = 0\right).$$
- The important fact to retain is that this index is easy to compute for reduced singularities and behaves well under reduction of singularities.

Gómez-Mont-Seade-Verjovsky indices

- Another invariant associated to a singularity is GSV index. In the smooth case, and with previous notations is

$$GSV_P(\mathcal{F}, \mathcal{C}) = \nu(q(0, z)).$$

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- It is an integer, which can also be computed (in the two-dimensional case) as

$$\dim_{\mathbb{C}} \left(\mathcal{O}/(f, A, B)\right),$$

whenever $\omega = A(y, z)dy + B(y, z)dz$ **[G]**.

Convergence of the weak separatrix

- The following relation holds between indices [**FM**]:

$$(d+2)^2 = CS_P(\mathcal{X},\mathcal{C}) + 2GSV_P(\mathcal{X},\mathcal{C}), \tag{2}$$

taking into account that in this family of examples, only one singularity (P) appears.

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This is a quite unexpected result, as generally, weak separatrices diverge. In this case the global structure of the foliation under study forces it to converge.

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¡MUCHAS GRACIAS!