The Borel map in the mixed Beurling setting

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Introduction

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$j_0^\infty : C^\infty(\mathbb{R}) \to \mathbb{C}^\mathbb{N}, \quad f \mapsto j_0^\infty(f) := (f^{(n)}(0))_{n\in\mathbb{N}}$ is called Borel map.

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Problem

For given $E \hookrightarrow C^\infty(\mathbb{R})$ find (large) F such that $F \subseteq j_0^\infty(E)$.

Ultradifferentiable classes

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$$\mathcal{E}^{\{M\}}(\mathbb{R}) := \Big\{ f \in C^\infty(\mathbb{R}) : \forall K \subset\subset \mathbb{R} \; \exists r > 0 : \|f\|^M_{K,r} < \infty \Big\},$$

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And the corresponding sequence space

$$\Lambda^{(M)} := \Big\{ \lambda = (\lambda_k)_k \in \mathbb{C}^{\mathbb{N}} : \forall r > 0 : \|\lambda\|^M_r := \sup_{k \in \mathbb{N}} \frac{|\lambda_k|}{r^k M_k} < \infty \Big\}.$$

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$$\Lambda^{(\omega)} := \left\{ \lambda = (\lambda_k)_k \in \mathbb{C}^{\mathbb{N}} : \forall r > 0 : \|\lambda\|^\omega_r := \sup_{k \in \mathbb{N}} \frac{|\lambda_k|}{e^{\phi^*_\omega(rk)/r}} < \infty \right\}.$$

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Definition

$$M \prec_{SV} N \;:\Leftrightarrow\; \exists C, s > 0 : \sup_{j \geq 1} \sup_{0 \leq i < j} \left(\frac{M_j}{s^j N_i} \right)^{\frac{1}{j-i}} \frac{1}{j} \sum_{k=j}^{\infty} \frac{N_{k-1}}{N_k} \leq C$$

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Theorem (Schmets, Valdivia 2003)

Let $M \prec N$ be weight sequences with $\liminf_{p\to\infty} \left(\frac{M_p}{p!}\right)^{1/p} > 0$. *Then*

$$\Lambda^{\{M\}} \subseteq j_0^\infty(\mathcal{E}^{\{N\}}(\mathbb{R})) \quad\Leftrightarrow\quad M \prec_{SV} N.$$

Theorem (Bonet, Meise, Taylor 1992)

Let ω, σ be weight functions. Then

$$\Lambda^{(\sigma)} \subseteq j_0^\infty(\mathcal{E}^{(\omega)}(\mathbb{R}))$$
$$\Leftrightarrow \;\; \kappa_\omega(r) := \int_1^\infty \frac{\omega(rt)}{t^2} dt = O(\sigma(r)) \text{ as } r \to \infty.$$

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Definition (Weight matrix)

A *weight matrix* is a one-parameter family of weight sequences $\mathfrak{M} = (M^{(x)})_{x>0}$ such that $M^{(x)} \leq M^{(y)}$ if $x \leq y$, and

$$\left(\frac{M_j^{(x)}}{j!}\right)^{1/j} \to \infty \text{ as } j \to \infty.$$

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A weight matrix $\mathfrak{M} = (M^{(x)})_{x>0}$ is said to have *moderate growth* if

$$\forall y > 0\ \exists x > 0 : \sup_{j+k\geq 1} \left(\frac{M^{(x)}_{j+k}}{M^{(y)}_j M^{(y)}_k}\right)^{\frac{1}{j+k}} < \infty, \qquad (\mathfrak{M}_{(mg)})$$

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to be *derivation closed* if

$$\forall y>0\ \exists x>0:\ \sup_{j\in\mathbb{N}}\left(\frac{M^{(x)}_{j+1}}{M^{(y)}_j}\right)^{\frac{1}{j+1}}<\infty, \qquad (\mathfrak{M}_{(dc)})$$

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and to be *non-quasianalytic* if

$$\forall x > 0 : \sum_{k=1}^{\infty} \frac{M^{(x)}_{k-1}}{M^{(x)}_k} < \infty. \qquad (\mathfrak{M}_{(nq)})$$

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Theorem (N., Rainer, Schindl 2022)

Let $\mathfrak{M}, \mathfrak{N}$ be weight matrices that are ordered with respect to their quotient sequences, i.e., $\mu^{(x)} \le \mu^{(y)}$ and $\nu^{(x)} \le \nu^{(y)}$ if $x \le y$. Then

$$\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty(\mathcal{E}^{(\mathfrak{N})}(\mathbb{R})) \iff \forall y > 0 \, \exists x > 0 : \; M^{(x)} \prec_{SV} N^{(y)}.$$

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This yields reduction to the (single weight sequence) Roumieu case.

A journey of (re)discovery and generalization

A parametrized Bonet–Meise–Taylor condition?

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For a weight sequence M, we define the *associated (pre-)weight function*

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BUT: For that we need moderate growth of $\mathfrak{M}$ and $\mathfrak{N}$.

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$$\begin{array}{ccccc} E & \xrightarrow{T} & F & \xleftarrow{R} & G \\ E' & \xleftarrow{T^t} & F' & \xrightarrow{R^t} & G' \end{array}$$

A functional analytic result by Bonet, Meise, Taylor

$$E \xrightarrow{T} F \xleftarrow{R} G$$

$$E' \xleftarrow{T^t} F' \xrightarrow{R^t} G'$$

Proposition

Let E, F, G be Fréchet–Schwartz spaces and let $T \in L(E, F)$ and $R \in L(G, F)$ have dense range. Assume that F' endowed with the initial topology with respect to $T^t : F' \to E'$ is bornological. Then the following conditions are equivalent:

- $R(G) \subseteq T(E)$.
- *If $B \subseteq F'$ is such that $T^t(B)$ is bounded in E', then $R^t(B)$ is bounded in G'.*

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Proposition

Suppose $(\Lambda^{(\mathfrak{N})})'$ endowed with the initial topology w.r.t. $(j_0^\infty)^t$ is bornological. Then the following conditions are equivalent:

- $\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty(\mathcal{E}^{(\mathfrak{N})}(\mathbb{R}))$.
- *If $B \subseteq (\Lambda^{(\mathfrak{N})})'$ is such that $(j_0^\infty)^t(B)$ is bounded in $\mathcal{E}^{(\mathfrak{N})}(\mathbb{R})'$, then $(\mathit{incl})^t(B)$ is bounded in $(\Lambda^{(\mathfrak{M})})'$.*

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$$\begin{aligned}
\mathcal{E}^{(\mathfrak{N})}(\mathbb{R})' &\overset{\mathcal{F}}{\cong} \{f \in \mathcal{H}(\mathbb{C}) : \ \exists A, k : \ |f(z)| \leq A e^{\omega_{N^{(1/k)}}(kz) + k|\mathrm{Im}(z)|}\} \\
&=: \mathcal{A}_{\Omega^+_{\mathfrak{N}}} \\
(\Lambda^{(\mathfrak{M})})' &\overset{S}{\cong} \{f \in \mathcal{H}(\mathbb{C}) : \ \exists A, k : \ |f(z)| \leq A e^{\omega_{N^{(1/k)}}(kz)}\} \\
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\end{aligned}$$

$$\begin{tikzcd}
\mathcal{E}^{(\mathfrak{N})}(\mathbb{R})' \arrow[d, "\mathcal{F}"] & (\Lambda^{(\mathfrak{N})})' \arrow[l, "(j_0^\infty)^t"'] \arrow[r, "\mathrm{incl}^t"] \arrow[d, "S"] & (\Lambda^{(\mathfrak{M})})' \arrow[d, "S"] \\
\mathcal{A}_{\Omega^+_{\mathfrak{N}}} & \mathcal{A}_{\Omega_{\mathfrak{N}}} \arrow[l, "\mathrm{incl}"] \arrow[r, "\mathrm{incl}"'] & \mathcal{A}_{\Omega_{\mathfrak{M}}}
\end{tikzcd}$$

Proposition

Suppose $\mathcal{A}_{\Omega_{\mathfrak{N}}}$ endowed with the trace topology w.r.t. $\mathcal{A}_{\Omega^+_{\mathfrak{N}}}$ is bornological. Then the following conditions are equivalent:

(i) $\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty(\mathcal{E}^{(\mathfrak{N})}(\mathbb{R}))$.

(ii) *If $B \subseteq \mathcal{A}_{\Omega_{\mathfrak{N}}}$ is such that B is bounded in $\mathcal{A}_{\Omega^+_{\mathfrak{N}}}$, then B is bounded in $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.*

Proposition

Suppose $\mathcal{A}_{\Omega_{\mathfrak{N}}}$ *endowed with the trace topology w.r.t.* $\mathcal{A}_{\Omega^+_{\mathfrak{N}}}$ *is bornological. Then the following conditions are equivalent:*

(i) $\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty(\mathcal{E}^{(\mathfrak{N})}(\mathbb{R}))$.

(ii) *If* $B \subseteq \mathcal{A}_{\Omega_{\mathfrak{N}}}$ *is such that* B *is bounded in* $\mathcal{A}_{\Omega^+_{\mathfrak{N}}}$, *then* B *is bounded in* $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.

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(ii) *If $B \subseteq \mathcal{A}_{\Omega_{\mathfrak{N}}}$ is such that B is bounded in $\mathcal{A}_{\Omega^+_{\mathfrak{N}}}$, then B is bounded in $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.*

(ii) reads as: Suppose that for all $f \in B$ there exist constants C_f and k_f such that

$$|f(z)| \leq C_f e^{\omega_{N^{(1/k_f)}}(k_f z)},$$

Proposition

Suppose $\mathcal{A}_{\Omega_{\mathfrak{N}}}$ endowed with the trace topology w.r.t. $\mathcal{A}_{\Omega_{\mathfrak{N}}^+}$ is bornological. Then the following conditions are equivalent:

(i) $\Lambda^{(\mathfrak{M})} \subseteq j_0^\infty(\mathcal{E}^{(\mathfrak{N})}(\mathbb{R}))$.

(ii) *If $B \subseteq \mathcal{A}_{\Omega_{\mathfrak{N}}}$ is such that B is bounded in $\mathcal{A}_{\Omega_{\mathfrak{N}}^+}$, then B is bounded in $\mathcal{A}_{\Omega_{\mathfrak{M}}}$.*

(ii) reads as: Suppose that for all $f \in B$ there exist constants C_f and k_f such that

$$|f(z)| \leq C_f e^{\omega_{N^{(1/k_f)}}(k_f z)},$$

and there are uniform constants C and k such that for all $f \in B$

$$|f(z)| \leq C e^{\omega_{N^{(1/k)}}(kz)+k|\mathrm{Im}(z)|}.$$

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Then one needs to be able to conclude the exitence of uniform constants $\tilde{C}$ and $\tilde{k}$ such that for all $f \in B$

$$|f(z)| \leq \tilde{C} e^{\omega_{M^{(1/\tilde{k})}}(\tilde{k} z)}.$$

A Phragmén Lindelöf Theorem

Let $f \in \mathcal{H}(\mathbb{C})$ and

$$\frac{\sup_{|z|=r} \log|f(z)|}{r} \overset{r\to\infty}{\to} 0, \qquad \int_{-\infty}^{\infty} \frac{\max(0, \log|f(t)|)}{1+t^2}\,dt < \infty. \quad (1)$$

Then

$$\log|f(z)| \le \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log|f(t)|}{(t-x)^2+y^2}\,dt.$$

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Corollary

Let $f \in \mathcal{A}_{\Omega_{\mathfrak{N}}}$ (which implies (1)!), and assume $f \in \mathcal{A}_{\Omega_{\mathfrak{N}}^+}$, i.e. there exist C, k such that $|f(z)| \leq C e^{\omega_{N^{(1/k)}}(kz) + k|y|}$. Then

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we are done!

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Theorem (N., Rainer, Schindl 2022)

Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are weight matrices, and assume that $\mathfrak{N}$ is derivation closed. Then the following conditions are equivalent:

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Rediscovery

- Langenbruch showed in 1994 an unparametrized version,
- Carleson showed in 1961 an unparametrized version (in disguise).

Special cases

Denjoy–Carleman classes

Theorem (N., Rainer, Schindl 2022)

Let M, N weight sequences, $M \leq CN$, and N derivation closed, with $\left(\frac{M_k}{k!}\right)^{1/k} \to \infty$ and $\left(\frac{N_k}{k!}\right)^{1/k} \to \infty$. Then the following are equivalent:

- $\Lambda^{(M)} \subseteq j_0^\infty(\mathcal{E}^{(N)}(\mathbb{R}))$.
- $\Lambda^{\{M\}} \subseteq j_0^\infty(\mathcal{E}^{\{N\}}(\mathbb{R}))$.
- *There is $C > 0$ such that $P_N(it) \leq \omega_M(Ct) + C$ for all $t > 0$.*
- $M \prec_{SV} N$.

If M has moderate growth, then the conditions are also equivalent to

- *There is $C > 0$ such that $\kappa_N(t) = O(\omega_M(t))$ as $t \to \infty$.*
- $\sup_{j\geq 1} \frac{M_j}{jM_{j-1}} \sum_{k\geq j} \frac{N_k}{N_{k-1}} < \infty$.

Braun–Meise–Taylor classes

Theorem (N., Rainer, Schindl 2022)

Let ω, σ be weight functions satisfying $\omega(t) = o(t)$, $\sigma(t) = o(t)$ as $t \to \infty$ and let $\Omega = (W^{(x)})_{x>0}$, $\Sigma = (S^{(x)})_{x>0}$ be the associated weight matrices. Then the following conditions are equivalent:

- $\Lambda^{(\sigma)} \subseteq j_0^\infty(\mathcal{E}^{(\omega)}(\mathbb{R}))$.
- $\Lambda^{\{\sigma\}} \subseteq j_0^\infty(\mathcal{E}^{\{\omega\}}(\mathbb{R}))$.
- $\kappa_\omega(t) = O(\sigma(t))$ *as* $t \to \infty$.
- *For all* $y > 0$ *there is* $x > 0$ *such that* $S^{(x)} \prec_{SV} W^{(y)}$.
- *For all* $y > 0$ *there is* $x > 0$ *such that* $P_{W^{(y)}}(it) \le \omega_{S^{(x)}}(Ct) + C$ *for all* $t > 0$.
- *There are* $x, y > 0$ *such that* $\kappa_{W^{(y)}}(t) = O(\omega_{S^{(x)}}(t))$ *as* $t \to \infty$.

Outlook

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- Prove analogous statements for the jet mapping

$$j_K^\infty : C^\infty(\mathbb{R}^d) \to C(K)^{\mathbb{N}^d}, \quad f \mapsto (f^{(\alpha)}|_K)_{\alpha \in \mathbb{N}^d}.$$

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- Find sharper results, i.e.

$$\Lambda^{(\mathfrak{M})} \subsetneq G \subseteq j_0^\infty(\mathcal{E}^{(\mathfrak{N})}(\mathbb{R}))$$

by investigating the diagram

$$\begin{tikzcd}
\mathcal{E}^{(\mathfrak{N})}(\mathbb{R})' \arrow[d, "\mathcal{F}"] & (\Lambda^{(\mathfrak{N})})' \arrow[l, "(j_0^\infty)^t"'] \arrow[r, "\mathrm{incl}^t"] \arrow[d, "S"] & G' \arrow[d] \\
\mathcal{A}_{\Omega_{\mathfrak{N}}^+} & \mathcal{A}_{\Omega_{\mathfrak{N}}} \arrow[l, "\mathrm{incl}"] \arrow[r, "?"'] & \mathcal{A}_G
\end{tikzcd}$$

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