Global regularity in ultradifferentiable spaces for non hypoelliptic PDE

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Workshop on Functional and Complex Analysis

Valladolid, 20 – 23 June 2022

Definition

A linear operator A on $\mathcal{S}'(\mathbb{R}^N)$ is globally regular if

$$u \in \mathcal{S}'(\mathbb{R}^N), \quad Au \in \mathcal{S}(\mathbb{R}^N) \implies u \in \mathcal{S}(\mathbb{R}^N).$$

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A polynomial $a(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is globally hypo-elliptic if

$$\lim_{|x|+|\xi|\to\infty} \frac{\left|\nabla a(x,\xi)\right|}{1+\left|a(x,\xi)\right|} = 0.$$

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Theorem (Tulovskiĭ and Šubin, 1973)

If $a(x, \xi) = \sum_{|\alpha|+|\beta|\leq m} x^\alpha \xi^\beta$ *is globally hypo-elliptic, then*

$$a(x, D) = \sum_{|\alpha|+|\beta|\leq m} x^\alpha D_x^\beta$$

is a globally regular operator.

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Global hypo-ellipticity regards both x and ξ. It is not necessary for global regularity.

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1. Are there necessary and sufficient conditions for global regularity of an operator $A : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$?
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- [E. Buzano, 2022] Necessary and sufficient conditions for global regularity of ordinary differential operators of order 2 with polymonial coefficients of any degree.

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$$\left(D_y - \frac{1}{2}x\right)^2 + \left(D_x + \frac{1}{2}y\right)^2$$

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- [C. Boiti, D. Jornet, A. O., 2017], [C. Mele, A. O., 2021] Global regularity in ultradifferentiable spaces for non hypoelliptic operators.

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Definition

A weight function (in the sense of Braun-Meise-Taylor) is a continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ such that:

- (α) there exists $K \geq 1$ such that $\omega(2t) \leq K(1 + \omega(t))$ for every $t \geq 0$;
- (β) $\omega(t) = o(t)$ as $t \to \infty$;
- (γ) there exist $a \in \mathbb{R}$, $b > 0$ such that $\omega(t) \geq a + b \log(1 + t)$, for every $t \geq 0$;
- (δ) $\varphi_\omega(t) = \omega \circ \exp(t)$ is a convex function.

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As usual, we extend ω to all $\mathbb{R}$ by defining $\omega(t) = \omega(|t|)$.

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in the sense that, for $x \in \mathbb{R}^N$,

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Let Ω, Σ be as before. We define $\mathcal{S}_\Omega^\Sigma(\mathbb{R}^N)$ as the space of all functions $f \in L^1(\mathbb{R}^N)$ such that $f, \hat{f} \in C^\infty(\mathbb{R}^N)$ and

$$\|\exp(\lambda\Omega) D^\alpha f\|_\infty < \infty, \text{ for each } \lambda > 0, \ \alpha \in \mathbb{N}_0^N,$$

$$\|\exp(\lambda\Sigma) D^\alpha \hat{f}\|_\infty < \infty, \text{ for each } \lambda > 0, \ \alpha \in \mathbb{N}_0^N.$$

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In the case when some of them coincide we can put together the corresponding variables, in the following sense: if for instance $\omega_2 = \omega_1$, we can choose

$$\Omega = \omega_1(x_1) + \omega_1(x_2) + \omega_3(x_3) + \cdots + \omega_N(x_N),$$

or

$$\Omega = \omega_1(|(x_1, x_2)|) + \omega_3(x_3) + \cdots + \omega_N(x_N),$$

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In particular, if $\omega_1 = \cdots = \omega_N = \sigma_1 = \cdots = \sigma_N := \omega$, the space $\mathcal{S}_\Omega^\Sigma(\mathbb{R}^N)$ coincides with the space $\mathcal{S}_\omega$ as defined in [Björck, 1966].

Basic properties

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- Consider the short-time Fourier transform

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Let $g \in \mathcal{S}^{\Sigma}_{\Omega}(\mathbb{R}^N)$, $g \neq 0$. Then for $f \in (\mathcal{S}^{\Sigma}_{\Omega})'(\mathbb{R}^N)$ the following conditions are equivalent:

(1) $f \in \mathcal{S}^{\Sigma}_{\Omega}(\mathbb{R}^N)$;

(2) For each $\lambda > 0$ there exists a constant $C_\lambda > 0$ so that

$$|V_g f(x,\xi)| \leq C_\lambda \exp(-\lambda(\Omega(x) + \Sigma(\xi))),$$

for each $(x,\xi) \in \mathbb{R}^{2N}$.

Young conjugates

We recall that

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We have the Young conjugates $\varphi^*_{\omega_j}$ and $\varphi^*_{\sigma_j}$ of φ_{ω_j} and φ_{σ_j} respectively, and we define

$$\Phi^*_\Omega(y) = \varphi^*_{\omega_1}(y_1) + \cdots + \varphi^*_{\omega_N}(y_N)$$

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for $y \in \mathbb{R}^N$.

Theorem: Equivalent seminorms in $\mathcal{S}_{\Omega}^{\Sigma}(\mathbb{R}^N)$

Let $1 \leq p, q \leq \infty$. Then the following conditions are equivalent:

(1) $f \in \mathcal{S}_{\Omega}^{\Sigma}(\mathbb{R}^N)$, i.e., $\forall \lambda > 0,\ \alpha \in \mathbb{N}_0^N$

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(5) (a) $\forall \lambda > 0$, $\beta \in \mathbb{N}_0^N$, there exists $C_{\beta,\lambda} > 0$ such that $\forall \alpha \in \mathbb{N}_0^N$

$$\left\|\exp\left(-\lambda\Phi_\Sigma^*\left(\frac{\alpha}{\lambda}\right)\right)x^\beta D^\alpha f\right\|_p \leq C_{\beta,\lambda};$$

(b) $\forall \mu > 0$, $\alpha \in \mathbb{N}_0^N$, there exists $C_{\alpha,\mu} > 0$ such that $\forall \beta \in \mathbb{N}_0^N$

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(6) $\forall \lambda, \mu > 0$ there exists $C_{\mu,\lambda} > 0$ such that $\forall \alpha, \beta \in \mathbb{N}_0^N$

$$\left\|\exp\left(-\lambda \Phi_\Sigma^*\left(\tfrac{\alpha}{\lambda}\right) - \mu \Phi_\Omega^*\left(\tfrac{\beta}{\mu}\right)\right) x^\beta D^\alpha f\right\|_p \le C_{\mu,\lambda}.$$

Wigner type transform

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$$\begin{array}{ccc} \mathrm{W}(Au, Au) & = & \mathrm{W}(f, f) \\ \parallel & & \parallel \\ B\left(\mathrm{W}(u, u)\right) & & F \end{array}$$

Wigner type transform

Given $f, g \in \mathcal{S}^{\Sigma}_{\Omega}(\mathbb{R}^N)$, the Wigner transform of f, g is given by

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If A is a (1 variable) linear operator with polynomial coefficients, B is a (2 variables) linear operator (with polynomial coefficients).

Wigner type transform

Definition

Given $u \in \mathcal{S}^{\Sigma}_{\Omega}(\mathbb{R}^{2N})$, we define $\mathrm{Wig}[u] : \mathbb{R}^{2N} \to \mathbb{C}$ as

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We consider an operator with polynomial coefficients

$$P(x,y,D_x,D_y) = \sum_{|\alpha+\beta+\gamma+\mu|\leq m} c_{\alpha\beta\gamma\mu} x^{\alpha} y^{\beta} D_x^{\gamma} D_y^{\mu},$$

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$$\mathrm{Wig}[Pu] = \boxed{??}\, \mathrm{Wig}[u]$$

Let

$$R_j : \mathcal{S}_\Omega^\Sigma(\mathbb{R}^{2N}) \to \mathcal{S}_\Omega^\Sigma(\mathbb{R}^{2N}), \quad j = 1, \ldots, N,$$

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If $\mathbf{R}$ and $\mathbf{T}$ are vectors of operators, and $a, b \in \mathbb{C}$ we can consider

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and then it makes sense to write $(a\mathbf{R} + b\mathbf{T})^\gamma$ for a multi index γ.

In particular, for $\mathbb{R}^{2N} \ni (x, y)$ we consider

$$
\begin{aligned}
\mathbf{M_f} F(x,y) &= (x_1 F(x,y), \ldots, x_N F(x,y)),\\
\mathbf{M_s} F(x,y) &= (y_1 F(x,y), \ldots, y_N F(x,y)),\\
\mathbf{D_f} F(x,y) &= (D_{x_1} F(x,y), \ldots, D_{x_N} F(x,y)),\\
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$$P(x, y, D_x, D_y) = \sum_{|\alpha+\beta+\gamma+\mu| \leq m} c_{\alpha\beta\gamma\mu} x^\alpha y^\beta D_x^\gamma D_y^\mu,$$

can be written as

$$P = P(\mathbf{M_f}, \mathbf{M_s}, \mathbf{D_f}, \mathbf{D_s}) = \sum_{|\alpha+\beta+\gamma+\mu| \leq m} c_{\alpha\beta\gamma\mu} \mathbf{M_f}^\alpha \mathbf{M_s}^\beta \mathbf{D_f}^\gamma \mathbf{D_s}^\mu$$

Theorem

Let P be a linear PDO with polynomial coefficients as before. Then for each $u \in \mathcal{S}(\mathbb{R}^{2N})$, the following formula holds:

$$\mathrm{Wig}\left[P\left(\mathbf{M_f}, \mathbf{M_s}, \mathbf{D_f}, \mathbf{D_s}\right) u\right] = \widetilde{P}(\mathbf{M_f}, \mathbf{M_s}, \mathbf{D_f}, \mathbf{D_s})\, \mathrm{Wig}[u],$$

for

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Observe that $\widetilde{P}$ is still and operator with polynomial coefficients.

Wigner type transform and operators

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What about global regularity?

Global regularity

$$\mathrm{Wig}\left[P\left(\mathbf{M_f}, \mathbf{M_s}, \mathbf{D_f}, \mathbf{D_s}\right) u\right] = \widetilde{P}(\mathbf{M_f}, \mathbf{M_s}, \mathbf{D_f}, \mathbf{D_s})\, \mathrm{Wig}[u]$$

Definition

- A linear operator P on $\mathcal{S}'(\mathbb{R}^{2N})$ is $\mathcal{S}$-globally regular if

$$u \in \mathcal{S}'(\mathbb{R}^{2N}), \quad Pu \in \mathcal{S}(\mathbb{R}^{2N}) \implies u \in \mathcal{S}(\mathbb{R}^{2N}).$$

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- If P has globally hypoelliptic symbol, P is globally regular.

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- If P has globally hypoelliptic symbol, P is globally regular.
- The symbol of $\widetilde{P}$ is never globally hypoelliptic.
- P is $\mathcal{S}$-globally regular $\iff$ $\widetilde{P}$ is $\mathcal{S}$-globally regular.
- What about $\mathcal{S}_\Omega^\Sigma$-globally regularity?

Action of Wigner type transform on $\mathcal{S}_{\Omega}^{\Sigma}$

$$\mathrm{Wig}[u](x,\xi) = \int_{\mathbb{R}^N} \exp(-it\xi) u\left(x + \frac{t}{2}, x - \frac{t}{2}\right)\, dt, \quad u \in \mathcal{S}_{\Omega}^{\Sigma}(\mathbb{R}^{2N}).$$

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$$\Omega(x,y) = \omega_{1,1}(x_1) + \cdots + \omega_{1,N}(x_N) + \omega_{2,1}(y_1) + \cdots + \omega_{2,N}(y_N),$$
$$\Sigma(\xi,\eta) = \sigma_{1,1}(\xi_1) + \cdots + \sigma_{1,N}(\xi_N) + \sigma_{2,1}(\eta_1) + \cdots + \sigma_{2,N}(\eta_N).$$

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Then

$$\mathrm{Wig} : \mathcal{S}_{\Omega}^{\Sigma}(\mathbb{R}^{2N}) \to \mathcal{S}_{\Omega_1}^{\Sigma_1}(\mathbb{R}^{2N});$$

where

$$\Omega_1(x,y) = \omega_{1,1}(x_1) + \cdots + \omega_{1,N}(x_N) + \sigma_{2,1}(y_1) + \cdots + \sigma_{2,N}(y_N),$$
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Action of Wigner type transform on $\mathcal{S}_\Omega^\Sigma$

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Then

$$\mathrm{Wig} : \mathcal{S}_\Omega^\Sigma(\mathbb{R}^{2N}) \to \mathcal{S}_{\Omega_1}^{\Sigma_1}(\mathbb{R}^{2N});$$
$$\mathrm{Wig} : (\mathcal{S}_\Omega^\Sigma)'(\mathbb{R}^{2N}) \to \left(\mathcal{S}_{\Omega_1}^{\Sigma_1}\right)'(\mathbb{R}^{2N});$$
$$\mathrm{Wig}^{-1} : \mathcal{S}_\Omega^\Sigma(\mathbb{R}^{2N}) \to \mathcal{S}_{\Omega_1}^{\Sigma_1}(\mathbb{R}^{2N});$$
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$$\mathrm{Wig}\left[P\left(\mathbf{M_f}, \mathbf{M_s}, \mathbf{D_f}, \mathbf{D_s}\right) u\right] = \widetilde{P}(\mathbf{M_f}, \mathbf{M_s}, \mathbf{D_f}, \mathbf{D_s})\, \mathrm{Wig}[u],$$

Global regularity in $\mathcal{S}_\Omega^\Sigma$

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Theorem [C. Mele, A. O.]

Let P be a linear PDO with polynomial coefficients and let Ω, Σ, Ω_1, Σ_1 be as before. P is $\mathcal{S}_\Omega^\Sigma$-regular $\Longleftrightarrow$ $\widetilde{P}$ is $\mathcal{S}_{\Omega_1}^{\Sigma_1}$-regular.

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Theorem [C. Mele, A. O.]

Let P be a linear PDO with polynomial coefficients and let Ω, Σ, Ω_1, Σ_1 be as before. P is $\mathcal{S}^{\Sigma}_{\Omega}$-regular $\iff$ $\widetilde{P}$ is $\mathcal{S}^{\Sigma_1}_{\Omega_1}$-regular.

More general transformations ban be used instead of Wigner type transform, namely, transforms in the Cohen class

$$Q[u] = \kappa \star Wig[u]$$

for kernels κ of the kind

$$\kappa(x, y) = \mathcal{F}^{-1}_{\substack{\xi \to x \\ \eta \to y}}\left[\exp\left(-i \sum_{j=1}^{N} p_j(\xi_j, \eta_j)\right)\right],$$

where p_j, $j = 1, \ldots, N$ are polynomials in $\mathbb{R}^2$ of any order, with coefficients in $\mathbb{R}$.

Examples

$$0 \neq p(z,\zeta) = \sum_{|\alpha+\beta|\leq m} c_{\alpha\beta} z^\alpha \zeta^\beta, \quad c_{\alpha\beta} \in \mathbb{C}, \quad z,\zeta \in \mathbb{R}^N.$$

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The following operators are $\mathcal{S}^\Sigma_\Omega$-regular for every Ω and Σ:

$$P_1 = \sum_{|\alpha+\beta|\leq m} c_{\alpha\beta} \left(\frac{x_1+y_1}{2} + R_1(D_{x_1}+D_{y_1})\right)^{\alpha_1} \cdots \left(\frac{x_N+y_N}{2} + R_N(D_{x_N}+D_{y_N})\right)^{\alpha_N}$$

$$\left(\frac{D_{x_1}-D_{y_1}}{2} + T_1(y_1-x_1)\right)^{\beta_1} \cdots \left(\frac{D_{x_N}-D_{y_N}}{2} + T_N(y_N-x_N)\right)^{\beta_N};$$

$$P_2 = \sum_{|\alpha+\beta|\leq m} c_{\alpha\beta} \left(x_1 - \frac{D_{y_1}}{2} - R_1(D_{x_1})\right)^{\alpha_1} \cdots \left(x_N - \frac{D_{y_N}}{2} - R_N(D_{x_N})\right)^{\alpha_N}$$

$$\left(x_1 + \frac{D_{y_1}}{2} - R_1(D_{x_1})\right)^{\beta_1} \cdots \left(x_N + \frac{D_{y_N}}{2} - R_N(D_{x_N})\right)^{\beta_N};$$

$$P_3 = \sum_{|\alpha+\beta|\leq m} c_{\alpha\beta} \left(\frac{D_{x_1}}{2} + y_1 - T_1(D_{y_1})\right)^{\alpha_1} \cdots \left(\frac{D_{x_N}}{2} + y_N - T_N(D_{y_N})\right)^{\alpha_N}$$

$$\left(\frac{D_{x_1}}{2} - y_1 + T_1(D_{y_1})\right)^{\beta_1} \cdots \left(\frac{D_{x_N}}{2} - y_N + T_N(D_{y_N})\right)^{\beta_N}.$$

where R_j and T_j are arbitrary polynomials with real coefficients.

Examples

The operators in $\mathbb{R}^2$

$$\left(D_x + \frac{1}{2}y\right)^2 + \left(D_y - \frac{1}{2}x\right)^2 \qquad \text{(Twisted Laplacian)}$$

are $\mathcal{S}^{\sigma_1 \oplus \sigma_2}_{\omega_1 \oplus \omega_2}$-regular, for every weight functions ω_1, ω_2, σ_1, σ_2.

Examples

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$$\left(D_x + \frac{1}{2}y\right)^{2h} + \left(D_y - \frac{1}{2}x\right)^{2k} \qquad \text{with } h, k \in \mathbb{N}$$

are $\mathcal{S}^{\sigma_1 \oplus \sigma_2}_{\omega_1 \oplus \omega_2}$-regular, for every weight functions ω_1, ω_2, σ_1, σ_2.

Examples

The operators in $\mathbb{R}^2$

$$\left(D_x + \frac{1}{2}y\right)^2 + \left(D_y - \frac{1}{2}x\right)^2 \qquad \text{(Twisted Laplacian)}$$

$$\left(D_x + \frac{1}{2}y\right)^{2h} + \left(D_y - \frac{1}{2}x\right)^{2k} \qquad \text{with } h, k \in \mathbb{N}$$

$$\left(x - \frac{1}{2}D_y + Q(D_x)\right)^2 + \left(y + \frac{1}{2}D_x + R(D_y)\right)^2$$

$$(x - D_y + Q(D_x))^2 + (y + R(D_y))^2$$

for arbitrary differential operators $Q(D_x)$ and $R(D_y)$ with real constant coefficients, are $\mathcal{S}^{\sigma_1 \oplus \sigma_2}_{\omega_1 \oplus \omega_2}$-regular, for every weight functions $\omega_1, \omega_2, \sigma_1, \sigma_2$.