Wild dynamics and periods for operators on Hilbert spaces

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Joint work with J.A. Conejero, F. Martínez-Giménez and F. Rodenas

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Sets of periods: Sharkovsky's theorem

Given a continuous map $f : X \to X$ on a topological space X, a point $x \in X$ is said to be periodic of period $p \in \mathbb{N}$ if $f^p(x) = x$ and $f^q(x) \neq x$ for any $q \in \mathbb{N}$ with $q < p$.

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Sharkovsky's theorem (1964)

Let $f : I \to I$, continuous on an interval $I \subset \mathbb{R}$, and consider the following total order in $\mathbb{N}$:

$$\begin{array}{ccccccccc}
3 & \prec & 5 & \prec & 7 & \prec & \dots & 2n+1 & \dots \\
3.\,2 & \prec & 5.\,2 & \prec & 7.\,2 & \prec & \dots & (2n+1).\,2 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
3.\,2^k & \prec & 5.\,2^k & \prec & 7.\,2^k & \prec & \dots & (2n+1).\,2^k & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & 2^n & \dots & \prec & 4 & \prec & 2 & \prec & 1
\end{array}$$

If f has periodic points of period m and $m \prec n$, then f has periodic points of period n.

Sets of periods: Sharkovsky's theorem

Moreover this order is optimal for interval maps, in the sense that, if $m \prec n$, one can find a continuous map $f : [0, 1] \to [0, 1]$ such that f has n-periodic points, but it doesn't admit any m-periodic point.

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Oleksandr Mykolaiovych Sharkovsky

Framework: Chaotic linear dynamics

Our framework will consist on linear maps (operators) $T : X \to X$ on an infinite-dimensional and separable complex Banach space. An operator $T : X \to X$ on X is called topologically transitive if, for any $U, V \subset X$ non-empty open sets there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$.

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The operator T is said to be Devaney chaotic if it is hypercyclic and admits a dense set of periodic points.

Sets of periods

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The set of periodic points is strongly related to the set of eigenvectors whose eigenvalue is an n-root of the unity. Actually, (Bonet, Martínez-Giménez, P. (2003)) we know that the set of periodic points of T is the vector space

$$\operatorname{span}\{x \in X \,/\, \exists n \in \mathbb{N}\,,\ \exists \lambda \in \mathbb{C} \;:\; \lambda^n = 1\,,\ Tx = \lambda x\}.$$

Roots of unity

Given $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ we say that λ is a primitive n-th root of 1 if $\lambda^n = 1$ and $\lambda^m \neq 1$ for $1 \leq m < n$. We denote by

$$\Lambda_n := \{\lambda \in \mathbb{C} \ : \ \lambda \text{ is a primitive } n\text{-th root of } 1\}$$

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For λ in the unit circle $\mathbb{T}$ and $\varepsilon > 0$, we denote by $I_{\lambda,\varepsilon}$ the open arc of the unit circle of length ε centered at λ.

Necessary conditions

For convenience, we only consider periodic vectors $x \in X \setminus \{0\}$.

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If $A \subset \mathbb{N}$ is a set of periods for an operator on a Banach space X, then A contains the least common multiple (lcm) of each pair of elements in A.

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If $A \subset \mathbb{N}$ is a set of periods for an operator on a Banach space X, then A contains the least common multiple (lcm) of each pair of elements in A.

Proof: Suppose that $A = \mathcal{P}(T)$ for some operator $T \in L(X)$. Let $n, m \in A$, $p = \operatorname{lcm}(m, n)$, and $x_1, x_2 \in X$ be n and m periodic vectors, respectively. These vectors can be expressed as a linear combination of eigenvectors corresponding to n-roots and m-roots of unity (BMP03):

$$x_1 = \sum_{i=1}^{k_n} \alpha_i y_i, \qquad \alpha_i \neq 0, \quad i = 1, \dots, k_n,$$

$$x_2 = \sum_{j=1}^{k_m} \beta_j z_j, \qquad \beta_j \neq 0, \quad j = 1, \dots, k_m.$$

Proof (continued):
Let $\lambda_i \in \Gamma_n$ be the eigenvalue of y_i, $i = 1, \dots k_n$, and let $\lambda'_j \in \Gamma_m$ be the eigenvalue of z_j, $j = 1, \dots, k_m$.

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We have that there are $n_i, m_j \in \mathbb{N}$, such that $\lambda_i \in \Lambda_{n_i}$ and $\lambda'_j \in \Lambda_{m_j}$, for $i = 1, \ldots, k_n$, $j = 1, \ldots, k_m$.

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Since x_1 is n-periodic and x_2 is m-periodic, we get $n = \operatorname{lcm}(n_1, \dots, n_{k_n})$ and $m = \operatorname{lcm}(m_1, \dots, m_{k_m})$. We define $x := \sum_{i=1}^{k_n} y_i + \sum_{j=1}^{k_m} z_j$, where we identify $y_i = z_j$ if $\lambda_i = \lambda'_j$ for some i and j. Finally, since $p = \operatorname{lcm}(n_1, \dots, n_{k_n}, m_1, \dots, m_{k_m})$, we deduce that x is a p-periodic vector for T and $p \in A$.

Sufficient conditions on the Hilbert space

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Proof: 1^{st} **step (Selection of suitable prime numbers):**
Fix $\theta \in]1, 4/3[\setminus \mathbb{Q}$. We select a non decreasing sequence $(n_m)_m$ of positive integers such that

$$\lim_m \frac{n_m}{m} = \theta/2 \quad \text{and} \quad \frac{m}{2} < n_m < \frac{2m}{3} \quad \text{for all } m > 4.$$

Proof 1^{st} **step (continued):**

By the Prime Number Theorem and the fact that $(n_m)_m$ is non decreasing and unbounded, for every $\varepsilon > 0$, there is $m_\varepsilon \in \mathbb{N}$ such that, if $m \geq m_\varepsilon$, then there is a prime number p satisfying $n_m < p < (1+\varepsilon)n_m$.

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Applying this result to $(1/k)_k$, $k \in \mathbb{N}$, we get an increasing sequence of positive integers $(m_k)_k$, with $m_1 > 4$, such that for each $k \in \mathbb{N}$ and for $m \in \mathbb{N}$ with $m_k \leq m < m_{k+1}$, there exists a prime number $p_{k,m}$ satisfying

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Now define

$$p_m := \begin{cases} 1 & \text{if } 1 \leq m < m_2 \\ p_{k,m} & \text{if } m_k \leq m < m_{k+1} \text{ for } k \geq 2 \end{cases}$$

Selection of prime numbers

Proof 1^{st} **step (continued):**
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This shows that $\eta_m := e^{2\pi i \frac{p_m}{m}} \in \Lambda_m$, $m \in \mathbb{N}$. Moreover, $\lim_m \eta_m = e^{\pi\theta i} \in \mathbb{T} \setminus e^{i\pi\mathbb{Q}}$.

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For convenience, sort positive integers not in A in increasing order and denote them as

$$A^c := \mathbb{N} \setminus A = \{j_1 < j_2 < \dots\}.$$

Selection of a Cantor set

Proof: 2^{nd} **step (Selection of a suitable Cantor set):**

Claim (proof not included): There exists a sequence $\{U_k\}_k$ with the following properties:
(1) $\cup_{k=1}^{\infty} U_k$ consists of a countable union of pairwise disjoint open arcs in $\mathbb{T}$, not sharing endpoints, where all endpoints have the form $e^{i\phi\pi}$ with $\phi \in]0,2[\setminus\mathbb{Q}$,

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Define now $U := \cup_{k=1}^{\infty} U_k$, which is an open subset of $\mathbb{T}$ and, by construction, $\mathbb{T} \setminus U$ does not have isolated points (i.e., it is a perfect set) and the primitive roots of the unity $\eta_m \in \mathbb{T} \setminus U$ if and only if $m \in A$.

Proof: 3^{rd} **step (Kalisch operator):**

To finish our proof we will need the Kalisch operator, that was used for the first time by Bayart and Grivaux (2005) in the context of chaotic properties in linear dynamics. More precisely, let $K : L^2[0, 2\pi] \to L^2[0, 2\pi]$ be defined as

$$Kf(\theta) = e^{i\theta} f(\theta) - \int_0^\theta ie^{it} f(t)dt, \quad \theta \in [0, 2\pi].$$

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It is well known (and easy to see) that, for any $\lambda \in \mathbb{T} \setminus \{1\}$, $\lambda = e^{i\beta}$ with $\beta \in]0,2\pi[$, we have

$$\ker(K - \lambda I) = \operatorname{span}(f_\lambda), \quad \text{where} \quad f_\lambda := \mathbf{1}_{[\beta,2\pi]}.$$

Proof: 3^{rd} **step (Continued):**

W.l.o.g. we suppose that $1 \in U$. Otherwise a suitable rotation of K does the job. We consider the Hilbert space $H := \overline{\text{span}}\{f_\lambda \ ; \ \lambda \in \mathbb{T} \setminus U\}$, which is K-invariant.

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Since $\mathbb{T} \setminus U$ is a compact perfect set and, by continuity of the map $\lambda \mapsto f_\lambda$, the set of eigenvectors of $T := K|_H$ associated to roots of unity is dense in H, T is chaotic (see, e.g., the book of Bayart and Matheron (2009)).

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Moreover, $\sigma_p(T) = \sigma(T) = \mathbb{T} \setminus U$. Given $m \in A$, we have that $\eta_m \in \Lambda_m \cap \sigma_p(T)$. Since all separable infinite dimensional Hilbert spaces are equivalent, we replace H by ℓ^2 from now on. Let $x \in \ell^2$ be an eigenvector of T associated to η_m, then x is m-periodic, and $A \subseteq \mathcal{P}(T)$.

Proof: 3^{rd} **step (Continued):**

On the other hand, if $m \in \mathcal{P}(T)$, proceeding as in part of the necessary conditions, we find a finite family $x_1, \ldots, x_k \in \ell^2$ of eigenvectors of T such that if $\lambda_1, \ldots, \lambda_k$ are the respective associated eigenvalues and $m_1, \ldots, m_k \in \mathbb{N}$ are so that $x_i \in \Lambda_{m_i}$, $i = 1, \ldots, k$, then $m = \mathrm{lcm}(m_1, \ldots, m_k)$.

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If $m_i \in A, i = 1, \ldots, k$, then the hypothesis on A imply that $m \in A$. If there was some $m_j \notin A$, then $\lambda_j \in U \cap \sigma_p(T)$ which is a contradiction.

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