Nonlinear conditions for ultradifferentiability

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Joint work with D. Nenning and G. Schindl

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Theorem (Joris 1982)

Let $f : M \to \mathbb{R}$ *be a function. If* f^2 *and* f^3 *are* C^∞*, then* f *is* C^∞*.*

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Remarks

- M is a C^∞-manifold. It is enough to consider $M = \mathbb{R}$ (Boman 1967).
- f (continuous) can take values in $\mathbb{C}$ or any complex function algebra (Duncan, Krantz, Parks 1985).
- The powers 2 and 3 can be replaced by any coprime p and q.
- More general nonlinear conditions: e.g. if $f^2 + f^3 \in C^\infty$ and $f^p \in C^\infty$ then $f \in C^\infty$.

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Division?

This is a kind of division result ($\frac{f^3}{f^2} = f$). There is a way to turn this naive approach into a rigorous proof (**fedja** on *MathOverflow*). This method works also for other regularity classes.

Joris's theorem for other regularity classes

- holomorphic
- real analytic
- polynomial
- Nash (C^∞ semialgebraic)
- ultradifferentiable classes, quasianalytic and non-quasianalytic

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Key for the proof

almost analytic extension and holomorphic approximation

For $\mathbf{M} = (M_k)_{k\geq 0}$ a positive sequence, $K \subseteq_{cp} \mathbb{R}^d$, and $\rho > 0$,

$$\|f\|^{\mathbf{M}}_{K,\rho} := \sup_{\alpha\in\mathbb{N}^d} \frac{\|f^{(\alpha)}\|_K}{\rho^{|\alpha|} M_{|\alpha|}}.$$

If $\mathfrak{M}$ is a totally ordered family of positive sequences and $U \subseteq \mathbb{R}^d$ is open,

$$\mathcal{E}^{\{\mathfrak{M}\}}(U) = \left\{ f \in C^\infty(U) : \ \forall K \subseteq_{cp} U \ \exists \mathbf{M} \in \mathfrak{M} \ \exists \rho > 0 : \|f\|^{\mathbf{M}}_{K,\rho} < \infty \right\}$$

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$\mathcal{E}^{\{\mathfrak{M}\}} =$ Roumieu , $\mathcal{E}^{(\mathfrak{M})} =$ Beurling , $\mathcal{E}^{[\mathfrak{M}]} =$ either case

Examples

- Denjoy–Carleman classes (e.g. $\mathcal{E}^{\{(k!^s)\}} =$ Gevrey G^s)
- Braun–Meise–Taylor classes (e.g. $\omega(t) = \max\{0, (\log t)^s\}$)
- intersection of all non-quasianalytic Gevrey classes

Assumptions

(1) Each $\mathbf{M} \in \mathfrak{M}$ is log-convex and $\mathbf{m} = (\frac{M_k}{k!})$ fulfills $m_0 = 1 \le m_1$ and $m_k^{1/k} \to \infty$.

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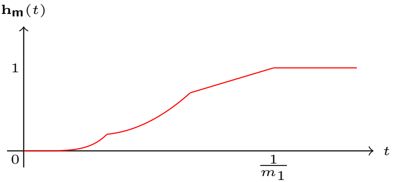
Associated functions: $\mathbf{h_m}(t) := \inf_{k\in\mathbb{N}} m_k t^k$ for $t > 0$ and $\mathbf{h_m}(0) := 0$.

$\mathbf{h_m}(t)$ is

- increasing,
- continuous,
- positive on $(0, \infty)$,
- 1 for large t,
- ∞-flat at 0.



Examples

$$\mathbf{h}_{(1)}|_{[0,1)} \equiv 0, \quad \mathbf{h}_{(\log(k+e)^{\delta k})}(t) \sim e^{-e^{\frac{1}{t^{1/\delta}}}}, \quad \mathbf{h}_{(k!^s)}(t) \sim e^{-\frac{1}{t^{1/s}}}$$

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If (1), (2), and (3) hold, we say that $\mathfrak{M}$ is admissible.

Admissible weight functions

Weight functions

A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ is a weight function if

- $\omega(2t) = O(\omega(t))$ as $t \to \infty$,
- $\omega(t) = o(t)$ as $t \to \infty$,
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Facts

- Braun–Meise–Taylor classes can be represented by $\mathcal{E}^{[\mathfrak{W}]}$, where $\mathfrak{W} = (\mathbf{W}^x)_{x>0}$ and $W_k^x = \exp(\frac{1}{x}\varphi^*(xk))$.
- $\mathfrak{W}$ is equivalent to an admissible weight if and only if ω is equivalent to a concave weight function.

Theorem (Thilliez 2020)

Let $\mathbf{M}$ *be such that* $\mathbf{m}$ *is log-convex and* $\sup_{j,k}(\frac{M_{j+k}}{M_j M_k})^{\frac{1}{j+k}} < \infty$*.*
Let $f : \mathbb{R} \to \mathbb{C}$ *be a function. If* $f^p, f^q \in \mathcal{E}^{\{\mathbf{M}\}}(\mathbb{R})$ *with* $\gcd(p, q) = 1$*, then* $f \in \mathcal{E}^{\{\mathbf{M}\}}(\mathbb{R})$*.*

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Let $\mathfrak{M}$ *be admissible. Let* $f : \mathbb{R}^d \to \mathbb{C}$ *be a function.*
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Uniformity: $f \mapsto (f^p, f^q)$ *is an injective map* $T : \mathbb{C}^{\mathbb{R}^d} \to \mathbb{C}^{\mathbb{R}^d} \times \mathbb{C}^{\mathbb{R}^d}$*.*
Its inverse $S : T(\mathbb{C}^{\mathbb{R}^d}) \to \mathbb{C}^{\mathbb{R}^d}$ *takes sets bounded in* $\mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^d) \times \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^d)$ *to sets bounded in* $\mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^d)$*.*

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Theorem (Nenning, R., Schindl 2021-22)

In the Braun–Meise–Taylor case, the result holds if and only if ω is equivalent to a concave weight function.

Theorem (Fürdös, Nenning, R. Schindl 2020; Dyn'kin 70ies)

Let $\mathfrak{M}$ *be admissible.* $f \in \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R})$ *if and only if for each compact interval* $I \subseteq \mathbb{R}$ *there exist* $\mathbf{M} \in \mathfrak{M}$*,* $\rho > 0$*, and (resp. for all* $\mathbf{M} \in \mathfrak{M}$ *and* $\rho > 0$ *there exists) an extension* $F \in C^1_c(\mathbb{C})$ *of* $f|_I$ *such that*

$$|\partial_{\overline{z}} F(z)| = |\tfrac{1}{2}(\partial_x + i\partial_y)F(z)| \le C\mathbf{h}_{\mathbf{m}}(\rho\, d(z,I)), \quad z \in \mathbb{C}.$$

Remark

If a class admits a description by almost holomorphic extension, then it has good stability properties.

Examples

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Holomorphic approximation

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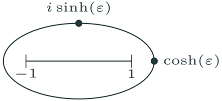

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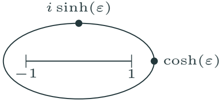

Theorem (HA)

Let $\mathfrak{M}$ be admissible.

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$$\|f_\varepsilon\|_{\Omega_\varepsilon} \le K, \ \|f - f_\varepsilon\|_{[-1,1]} \le c_1 \mathbf{h_m}(c_2\varepsilon), \ \textit{for all small } \varepsilon > 0. \quad (\dagger)$$

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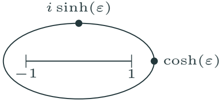

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2. *Let $f : [-1,1] \to \mathbb{C}$. If there exist $f_\varepsilon \in \mathcal{H}(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$ and* $\mathbf{M} \in \mathfrak{M}$ *such that* ($\dagger$)*, then $f \in \mathcal{E}^{\{\mathfrak{M}\}}((-1,1))$.*

The strategy

Let $f : [-1,1] \to \mathbb{C}$ such that $g := f^2, h := f^3 \in \mathcal{E}^{\{\mathfrak{M}\}}([-1,1])$.
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Step 1: Divide: for suitable $r_\varepsilon > 0$, $\varphi_\varepsilon \in C_c^\infty(\Omega_\varepsilon)$, $\varphi_\varepsilon|_{\Omega_{\varepsilon/2}} = 1$,

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Step 3: Apply HA2 after showing that for some $\mathbf{M}' \in \mathfrak{M}$

$$\|f_\varepsilon\|_{\Omega_\varepsilon} \le L, \ \|f - f_\varepsilon\|_{[-1,1]} \le d_1 \mathbf{h}_{\mathbf{m}'}(d_2 \varepsilon), \quad \text{for all small } \varepsilon > 0.$$

- **Polarization inequality:** For $d_v^k f(x) := (\frac{\partial}{\partial t})^k f(x+tv)|_{t=0}$,

$$\sup_{\|v\|\le 1} |d_v^k f(x)| \le \|f^{(k)}(x)\|_{L^k(\mathbb{R}^d,\mathbb{C})} \le (2e)^k \sup_{\|v\|\le 1} |d_v^k f(x)|.$$

- **Uniform unidirectional holomorphic approximation:**

$f : \mathbb{R}^d \to \mathbb{C}$ is of class $\mathcal{E}^{\{\mathfrak{M}\}}$ if and only if the functions

$$f_{x,v}(t) := f(x+tv), \quad t \in [-1,1],\ x \in \mathbb{R}^d,\ v \in \mathbb{S}^{d-1},$$

admit uniform holomorphic approximation, i.e., there exist $f_{x,v,\varepsilon} \in \mathcal{H}(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$ and $\mathbf{M} \in \mathfrak{M}$ such that

$$\|f_{x,v,\varepsilon}\|_{\Omega_\varepsilon} \le K, \quad \|f_{x,v} - f_{x,v,\varepsilon}\|_{[-1,1]} \le c_1 \mathbf{h_m}(c_2\varepsilon)$$

uniformly for all x, v and all small $\varepsilon > 0$.

From one to many variables

- **Polarization inequality:** For $d_v^k f(x) := (\frac{\partial}{\partial t})^k f(x+tv)|_{t=0}$,

$$\sup_{\|v\|\le 1} |d_v^k f(x)| \le \|f^{(k)}(x)\|_{L^k(\mathbb{R}^d,\mathbb{C})} \le (2e)^k \sup_{\|v\|\le 1} |d_v^k f(x)|.$$

- **Uniform unidirectional holomorphic approximation:**

$f : \mathbb{R}^d \to \mathbb{C}$ is of class $\mathcal{E}^{\{\mathfrak{M}\}}$ if and only if the functions

$$f_{x,v}(t) := f(x+tv), \quad t \in [-1,1],\ x \in \mathbb{R}^d,\ v \in \mathbb{S}^{d-1},$$

admit uniform holomorphic approximation, i.e., there exist $f_{x,v,\varepsilon} \in \mathcal{H}(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$ and $\mathbf{M} \in \mathfrak{M}$ such that

$$\|f_{x,v,\varepsilon}\|_{\Omega_\varepsilon} \le K, \quad \|f_{x,v} - f_{x,v,\varepsilon}\|_{[-1,1]} \le c_1 \mathbf{h_m}(c_2\varepsilon)$$

uniformly for all x, v and all small $\varepsilon > 0$.

Consequence: The result holds on arbitrary infinite dimensional Banach spaces and, more generally, on all convenient vector spaces.

More general nonlinear conditions

Let $f : (\mathbb{R}^d, 0) \to (\mathbb{K}, 0)$ be a continuous germ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). We know:

$$\Phi \circ f \in \mathcal{E}^{[\mathfrak{M}]} \implies f \in \mathcal{E}^{[\mathfrak{M}]} \qquad (\mathscr{P}_{[\mathfrak{M}]})$$

if $\Phi(t) = (t^p, t^q)$ with $\gcd(p, q) = 1$ and $\mathfrak{M}$ admissible.

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Problem

Characterize the *analytic* germs $\Phi : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$ with $(\mathscr{P}_{[\mathfrak{M}]})$.

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Necessary condition

Write $\Phi(t) = \sum a_k t^{n_k}$, where $a_k \in \mathbb{K}^n \setminus \{0\}$ and $\{n_1, n_2, \ldots\} \subseteq \mathbb{N}_{\geq 1}$ is the support of $\Phi(t)$.

If Φ has property $(\mathscr{P}_{[\mathfrak{M}]})$, then $\gcd(n_1, n_2, \ldots) = 1$.

In fact, if $n_k = p\ell_k$ for all k, then $\Psi(t) := \sum a_k t^{\ell_k}$ is convergent and $\Phi(t) = \Psi(t^p)$. There is a continuous germ $f \notin C^1$ with $\Phi \circ f$ analytic.

More general nonlinear conditions

We say that $\Phi, \Psi : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$ are *equivalent* if there are germs of analytic diffeomorphisms $u : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ and $v : (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ such that $u \circ \Phi \circ v = \Psi$.

Equivalent germs either both satisfy or do not satisfy $(\mathscr{P}_{[\mathfrak{M}]})$.

It is no restriction to assume that $\Phi_1(t) = t^p$ for some $p \in \mathbb{N}_{\ge 1}$.

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Theorem (Nenning, R., Schindl 2021-22)

Let $\Phi : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$ be analytic, $\Phi_1(t) = t^p$, and $\{n_1, n_2, \dots\}$ the support of the power series $\Phi(t)$. Let $\mathfrak{M}$ be admissible. Then Φ has property $(\mathscr{P}_{[\mathfrak{M}]})$ if and only if $\gcd(n_1, n_2, \dots) = 1$.

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Remark

In the C^∞-setting this is due to (Joris & Preissmann 1987): Here Φ is a C^∞-germ (with complex Taylor series if $\mathbb{K} = \mathbb{C}$). Then

$$(\forall f \in C^0 : \Phi \circ f \in C^\infty \implies f \in C^\infty) \iff \gcd(n_1, n_2, \ldots) = 1.$$

Idea of the proof

We may assume that $p \geq 2$ and $n = 2$: there are $\gamma_i \in \mathbb{R}$ such that the support of $(t^p, \varphi(t) := \gamma_2 \Phi_2(t) + \cdots + \gamma_n \Phi_n(t))$ has $\gcd = 1$.

Show that there is $q \in \mathbb{N}_{\geq 1}$ and analytic germs α_j at 0 such that

$$t^{1+pq} = \sum_{j=0}^{p-1} \alpha_j(t^p)\varphi(t)^j.$$

Thus we may apply the result for $\Phi(t) = (t^p, t^{1+pq})$.

Uniformity

In the scope of the theorem, the map $\Phi \circ f \mapsto f$ takes bounded sets in $\mathcal{E}^{[\mathfrak{M}]}$ to bounded sets in $\mathcal{E}^{[\mathfrak{M}]}$.

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Admissibility of $\mathfrak{M}$ is "close to necessary"

$(\mathscr{P}_{[\mathfrak{M}]})$ implies that $\mathcal{E}^{[\mathfrak{M}]}$ is inverse-closed.

If g is a continuous germ at $0 \in \mathbb{R}$ with $g(0) = 1$, then $f := 1/g$ and $h := f - 1$ are continuous germs and $h(0) = 0$. Let $\varphi(t) = \frac{1}{1+t} - 1 = \sum_{k \geq 1} (-1)^k t^k$. For any germ Φ which has φ as component the support satisfies $\gcd = 1$. If g is $\mathcal{E}^{[\mathfrak{M}]}$ then $\varphi \circ h = g - 1$ is $\mathcal{E}^{[\mathfrak{M}]}$. By the theorem, h and thus $f = 1/g$ is $\mathcal{E}^{[\mathfrak{M}]}$.

In particular, if $\mathcal{E}^{[\omega]}$ satisfies $\mathscr{P}_{[\omega]}$ for all Φ whose support has $\gcd = 1$, then ω is concave (up to equivalence of weight functions).

Examples

- $f^2 + f^3 \in \mathcal{E}^{[\mathfrak{M}]}$ and $f^p \in \mathcal{E}^{[\mathfrak{M}]}$ imply $f \in \mathcal{E}^{[\mathfrak{M}]}$.

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If the target is not a complex function algebra:

- In a Banach algebra with a non-zero nilpotent element x of order 2, $f(t) := \left\{ \begin{smallmatrix} x & \text{if } t\in\mathbb{Q}, \\ 0 & \text{if } t\notin\mathbb{Q}. \end{smallmatrix} \right\}$ is discontinuous, but $f^2 = f^3 = 0$.

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- Consider the algebra of all $A = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ with $\|\cdot\|_2$-norm.
If $a(t) := e^{-\frac{1}{t^2}}$, $a(0) := 0$, and $b(t) := |t|$ for $t \in \mathbb{R}$, then
$f := \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \in C^0 \setminus C^1$, but $f^2 = \left(\begin{smallmatrix} a^2 & 2ab \\ 0 & a^2 \end{smallmatrix}\right)$ and $f^3 = \left(\begin{smallmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{smallmatrix}\right)$ are C^∞.

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- Joris's Theorem is wrong for quaternion valued functions. $\mathbb{H} = \left\{\left(\begin{smallmatrix} z & w \\ -\overline{w} & \overline{z} \end{smallmatrix}\right) : z, w \in \mathbb{C}\right\}$. There is $f(t) := \left(\begin{smallmatrix} 0 & w(t) \\ -\overline{w}(t) & 0 \end{smallmatrix}\right) \in C^1 \setminus C^2$ such that $f^2 = -|w|^2 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $f^3 = |w|^2 \left(\begin{smallmatrix} 0 & -w \\ \overline{w} & 0 \end{smallmatrix}\right)$ are C^∞.

References

Duncan, J., Krantz, S. G., & Parks, H. R. (1985). Nonlinear conditions for differentiability of functions. *J. Analyse Math.*, *45*, 46–68.

Fürdös, S., Nenning, D. N., Rainer, A., & Schindl, G. (2020). Almost analytic extensions of ultradifferentiable functions. *J. Math. Anal. Appl.*, *481*(1). (doi:10.1016/j.jmaa.2019.123451)

Joris, H. (1982). Une $\mathcal{C}^\infty$-application non-immersive qui possède la propriété universelle des immersions. *Arch. Math. (Basel)*, *39*(3), 269–277.

Joris, H., & Preissmann, E. (1987). Pseudo-immersions. *Ann. Inst. Fourier (Grenoble)*, *37*(2), 195–221.

Nenning, D. N., Rainer, A., & Schindl, G. (2021). Nonlinear conditions for ultradifferentiability. *J. Geom. Anal.*, *31*, 12264–12287. (doi:10.1007/s12220-021-00718-w)

Nenning, D. N., Rainer, A., & Schindl, G. (2022). Nonlinear conditions for ultradifferentiability: a uniform approach. *J. Geom. Anal.*, *32*, Article number: 171. (doi:10.1007/s12220-022-00914-2)

Thilliez, V. (2020). Functions with ultradifferentiable powers. *Results Math.*, *75*, Article number: 79. (doi:10.1007/s00025-020-01205-4)