Uniformly ergodic measures on locally compact groups

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Joint work with Jorge Galindo and Enrique Jordá

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E is a Banach space.
$T : E \longrightarrow E$ a continuous and linear operator.

$$\text{Iterates: } T^n = T \circ \overset{\text{n-fold}}{\cdots} \circ T$$

$$\text{Cesàro means: } T_{[n]} = \frac{1}{n} \sum_{m=1}^{n} T^m$$

T is called **power bounded** if the set $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$.
T is called **(uniformly) mean ergodic** when $(T_{[n]})_n$ converges in the strong operator topology (resp. uniformly on bounded sets).

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$C_0(G)$ continuous functions vanishing at infinity.
$M(G) = C_0(G)^*$

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L_p spaces:

$$L_p(G) = \{f \, : \, \|f\|_p = \left(\int_G |f|^p dm_G \right)^{1/p} < \infty\}.$$

Convolution by a probability measure μ:

$$(\mu * f)(s) = \int_G f(x^{-1}s)d\mu(x), \quad f \in L_p(G), s \in G$$

Convolution operator: $\lambda_p(\mu) : L_p(G) \longrightarrow L_p(G)$,

$$[\lambda_p(\mu)](f) = \mu * f.$$

μ probability measure.

Theorem (Galindo, Jordá)

G amenable, $1 < p < \infty$
$\lambda_p(\mu)$ mean ergodic and power bounded.

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G Abelian.
$\lambda_1(\mu)$ uniformly mean ergodic $\iff$ 1 is isolated in $\sigma(\lambda_1(\mu))$ and $\overline{\langle S_\mu \rangle}$ compact.

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Problem: The relation between the uniform mean ergodicity of $\lambda_1(\mu)$ and $\lambda_1^0(\mu)$.

Theorem (Galindo, Jordá, R.)

G Abelian, μ probability measure. TFAE

(i) $\lambda_1(\mu)$ *is uniformly mean ergodic.*

(ii) $\lambda_1^0(\mu)$ *is uniformly mean ergodic.*

(iii) $\overline{\langle S_\mu \rangle}$ *is compact and* 1 *isolated in* $\sigma(\lambda_1(\mu))$.

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Sketch of the proof: (i) and (iii) are equivalent from earlier.

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$*$If G is compact, $L_1(G) = L_1^0(G) \oplus \mathbb{C}$ since $f = \left(f - \int f\right) + \int f$.
$L_1^0(G)$ invariant by $\lambda_1(\mu) \Rightarrow \lambda_1(\mu) = \lambda_1^0(\mu) \oplus I$. Done

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$*$ G not compact. $\lambda_1^0(\mu)$ is uniformly mean ergodic $\Rightarrow$ 1 isolated in $\sigma(\lambda_1^0(\mu)) \Rightarrow$ 1 isolated in $\sigma(\lambda_1(\mu)) \Rightarrow \lambda_1(\mu)$ is uniformly mean ergodic

Definition

A probability measure $\mu \in M(G)$ is:

(i) *adapted* if $\overline{\langle S_\mu \rangle} = G$

(ii) *strictly aperiodic* if it is adapted and every normal subgroup satisfying $S_\mu \subseteq xN$ for some $x \in G$ is actually $N = G$.

(iii) *spread-out* if μ^n is not singular for some $n \in \mathbb{N}$ (i.e. if there is $n \in \mathbb{N}$ such that there is no set $A \subseteq G$ with $\mu^n(A) = 1$ and $m_G(A) = 0$).

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Corollary

μ adapted ($\overline{\langle S_\mu \rangle} = G$)
$\lambda_1(\mu)$ mean ergodic $\iff$ G compact.

Definition

A probability measure μ is

(i) *ergodic (by convolutions)* if $\lim_n [\lambda_1^0(\mu)]_{[n]} f = 0$ for $f \in L_1^0(G)$.

(ii) *completely mixing* if $\lim_n [\lambda_1^0(\mu)]^n f = 0$ for $f \in L_1^0(G)$.

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(i) *uniformly ergodic* if $\lim_n \|[\lambda_1^0(\mu)]_{[n]}\| = 0$.

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Theorem

$\overline{\langle S_\mu \rangle}$ *compact* $\Rightarrow \mu_{[n]} \longrightarrow m_{\overline{\langle S_\mu \rangle}}$ *in* $\sigma(M(G), C_0(G))$.

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μ adapted $\Rightarrow G = \overline{\langle S_\mu \rangle} \Rightarrow \mu_{[n]} \longrightarrow m_G$

$$\mu_{[n]} * f \longrightarrow m_G * f = \int_G f \, dm_G = 0, \quad f \in L_1^0(G)$$

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Corollary (Reminder)

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Example

G Abelian, noncompact, μ adapted. Then μ is ergodic but $\lambda_1(\mu)$ is not mean ergodic.

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Example

G Abelian, compact, μ not adapted. Then μ is not ergodic but $\lambda_1(\mu)$ is mean ergodic.

(μ strictly aperiodic $\iff$ adapted and every normal subgroup satisfying $S_\mu \subseteq xN$ for some $x \in G$ is actually $N = G$.)

μ completely mixing $\Rightarrow$ μ strictly aperiodic.
G is compact or Abelian: μ strictly aperiodic $\Rightarrow$ μ completely mixing.

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Question: μ ergodic and strictly aperiodic $\Rightarrow$ μ completely mixing?

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True if:

- μ spread out (Glasner)
- G compact (Ito-Kawada)
- G is in [SIN] (Jaworski)
- μ^n and μ^{n+1} are not mutually singular for some $n \in \mathbb{N}$ (Foguel)

(μ strictly aperiodic $\iff$ adapted and every normal subgroup satisfying $S_\mu \subseteq xN$ for some $x \in G$ is actually $N = G$.)

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Question: μ uniformly ergodic and strictly aperiodic $\Rightarrow$ μ uniformly completely mixing?

(μ spread-out $\iff$ $\exists n \in \mathbb{N}$: $\nexists A \subseteq G$ with $\mu^n(A) = 1$ and $m_G(A) = 0$)

Theorem (Galindo, Jordá, R.)

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Sketch of the proof:

$*$ μ spread-out $\Rightarrow \exists n : \mu^n = \mu_s + \mu_a$, $\mu_a = f \cdot m_G$, $f > 0$, μ_s singular and positive.

$$\|\lambda_1^0(\mu^n) - \lambda_1^0(\mu_a)\| = \|\lambda_1^0(\mu_s)\| < 1.$$

$\lambda_1^0(\mu_a)$ is a compact operator $\Rightarrow \lambda_1^0(\mu)$ is quasicompact. Done.

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$\lambda_1^0(\mu_a)$ is a compact operator $\Rightarrow \lambda_1^0(\mu)$ is quasicompact. Done.

$*$ μ not spread-out $\Rightarrow$ (technical stuff) $\Rightarrow$ 1 not isolated in $\sigma(\lambda_1^0(\mu)) \Rightarrow \lambda_1^0(\mu)$ not uniformly mean ergodic.

(completely mixing $\iff$ $[\lambda_1^0(\mu)]^n f \longrightarrow 0$ for $f \in L_1^0(G)$)

Theorem (Galindo, Jordá, R.)

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Theorem (Yosida-Kakutani)

$T \in L(X)$ power bounded, quasicompact and without eigenvalues of modulus $1 \Rightarrow \|T^n\| \longrightarrow 0$.

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Sketch of the proof: μ completely mixing $\iff$ μ strictly aperiodic

$*$ μ uniformly ergodic $\Rightarrow$ μ spread-out $\Rightarrow$ $\lambda_1^0(\mu)$ is quasicompact.

$*$ μ strictly aperiodic $\Rightarrow$ (technical stuff) $\Rightarrow$ no eigenvalues of modulus 1.

Theorem

G Abelian and connected, μ adapted.
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Idea of proof: μ uniformly ergodic $\Rightarrow$ μ strictly aperiodic.

Example

G Abelian non-connected.
$\exists H$ maximal proper normal open subgroup. Define $\mu = 1_{xH} m_G$ for some $x \notin H$.
By normality: $H \subsetneq \langle xH \rangle$.
By maximality: $\overline{\langle S_\mu \rangle} = \overline{\langle xH \rangle} = G \Rightarrow \mu$ adapted.
From its definition μ is spread-out.
μ is adapted and spread-out, but it is not strictly aperiodic by construction.

Theorem

G Abelian, $\overline{\langle S_\mu \rangle}$ compact. TFAE

(i) $\lambda_1(\mu) : L_1(G) \longrightarrow L_1(G)$ *uniformly mean ergodic.*

(ii) $\lambda_1^0(\mu) : L_1^0(G) \longrightarrow L_1^0(G)$ *uniformly mean ergodic.*

(iii) $\lambda_1(\mu) : L_1(\overline{\langle S_\mu \rangle}) \longrightarrow L_1(\overline{\langle S_\mu \rangle})$ *uniformly mean ergodic.*

(iv) $\lambda_1^0(\mu) : L_1^0(\overline{\langle S_\mu \rangle}) \longrightarrow L_1^0(\overline{\langle S_\mu \rangle})$ *uniformly mean ergodic.*

(v) μ *uniformly ergodic on* $M(\overline{\langle S_\mu \rangle})$.

(vi) μ *spread-out on* $\overline{\langle S_\mu \rangle}$.

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(v) μ *uniformly ergodic on* $M(\overline{\langle S_\mu \rangle})$.

(vi) μ *spread-out on* $\overline{\langle S_\mu \rangle}$.

Example

H compact subgroup. $\mu = m_H$. Then μ uniformly ergodic on $L_1^0(\overline{\langle S_\mu \rangle})$ but not on $L_1^0(G)$.

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