Norm-attainment questions for nuclear operators and tensors

Óscar Roldán

A joint work with
Sheldon Dantas, Mingu Jung and Abraham Rueda Zoca

VNIVERSITAT DE VALÈNCIA

Workshop on Functional and Complex Analysis
23rd of June 2022, Valladolid

MICIU grant FPU17/02023.
MINECO & FEDER project MTM2017-83262-C2-1-P.

About the talk

1) The contents of this talk are from:

S. Dantas, M. Jung, Ó. Roldán, A. Rueda Zoca, Norm attaining tensors and nuclear operators, *Mediterranean Journal of Mathematics* **19** (1) (2022), Paper no. 38, 27pp.

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2) This talk is dedicated to my Ukrainian friends and colleagues, and in particular to the Kharkiv team who welcomed me so warmly during my stay there last year.

#StopThisWar
#StopPutin
#StandWithUkraine

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- $\mathcal{B}(X \times Y, Z)$: space of bilinear mappings from $X \times Y$ into Z.

The title

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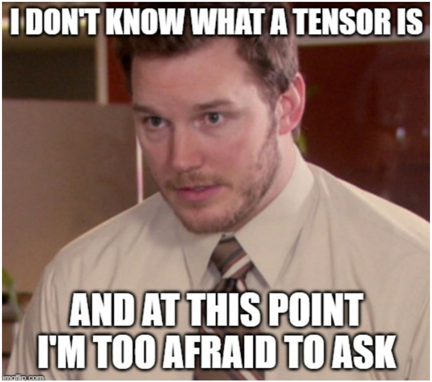
Tensor products



Picture from: https://www.reddit.com/r/physicsmemes/comments/

Tensor products

Figure: Useful definition of tensor, according to several sources.

Original picture from The Simpsons, season 5, episode 10. Obtained at: https://frinkiac.com/caption/S05E10/143909

Tensor products - Brief introduction

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V, W $\mathbb{K}$-vector spaces, $V \otimes W$ is a v. s. that can be formed as $A/\sim$, where:

$$A := \left\{ \sum_{k=1}^{n} \lambda_k (v_k, w_k) : \; n \in \mathbb{N}, \lambda_k \in \mathbb{K}, v_k \in V, w_k \in W \right\}$$

$$\begin{cases} (v_1 + v_2, w) \sim (v_1, w) + (v_2, w) \\ (v, w_1 + w_2) \sim (v, w_1) + (v, w_2) \end{cases} \qquad \begin{cases} \lambda(v, w) \sim (\lambda v, w) \\ \lambda(v, w) \sim (v, \lambda w) \end{cases}$$

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The elements are called **tensors**, and are denoted like this: $z = \sum_{k=1}^{n} \lambda_k v_k \otimes w_k$.

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Equivalently, $V \otimes W$ can be seen as the linear subspace of $\mathcal{B}(X \times Y, \mathbb{K})^{\#}$ (algebraic dual) spanned by these evaluation maps:

$$(v \otimes w)(A) = A(v, w), \quad A \in \mathcal{B}(X \times Y, \mathbb{K}), \, v \in V, \, w \in W.$$

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Universal property: if $\mu : V_1 \times V_2 \to V_1 \otimes V_2$ is the canonical mapping such that $\mu(v_1, v_2) = v_1 \otimes v_2$, then for any bilinear mapping $f \in \mathcal{B}(V_1 \times V_2, W)$, there is an unique linear mapping $\overline{f} : V_1 \otimes V_2 \to W$ with $\overline{f}(v_1 \otimes v_2) = f(v_1, v_2)$.

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- $(X\widehat{\otimes}_\pi Y)^* = \mathcal{B}(X \times Y, \mathbb{K}) = \mathcal{L}(X, Y^*) = \mathcal{L}(Y, X^*)$, where $\forall G : X \to Y^*$,

$$G\left(\sum_{n=1}^{\infty} x_n \otimes y_n\right) = \sum_{n=1}^{\infty} G(x_n)(y_n).$$

R. A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

Nuclear Operators

Figure: Nuclear operators (probably unrelated to the contents of this talk).

Original picture from The Simpsons, season 15, episode 12. Obtained at: https://frinkiac.com/caption/S15E12/95470

Nuclear Operators - Building them from tensors

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There is a **canonical operator** $J : X^*\widehat{\otimes}_\pi Y \to \mathcal{L}(X,Y)$ s.t. if $z = \sum_{n=1}^\infty x_n^* \otimes y_n$, then $J(z) = L_z$, where $L_z(x) = \sum_{n=1}^\infty x_n^*(x) y_n$ for $x \in X$.

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Operators that can be expressed in that form are called **nuclear**. The set of nuclear operators is $\mathcal{N}(X, Y)$ endowed with the **nuclear norm**:

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- Nuclear operators are limits of finite rank operators, and hence compact.
- $\mathcal{N}(X,Y)$ can be identified with $(X^* \widehat{\otimes}_\pi Y)/\mathrm{Ker}(J)$ isometrically.
- If X^* or Y has the approximation property, then $(X^* \widehat{\otimes}_\pi Y) = \mathcal{N}(X,Y)$.

Recall that a Banach space has the **approximation property** if $\forall K \subset X$ compact and $\forall \varepsilon > 0$, there exists $T \in \mathcal{F}(X,X)$ with $\|T(x) - x\| < \varepsilon$ for all $x \in K$.

- c_0, ℓ_p, $L_p(\mu)$, $C(K)$ and spaces with Shauder basis all have the A.P.

Norm-attainment

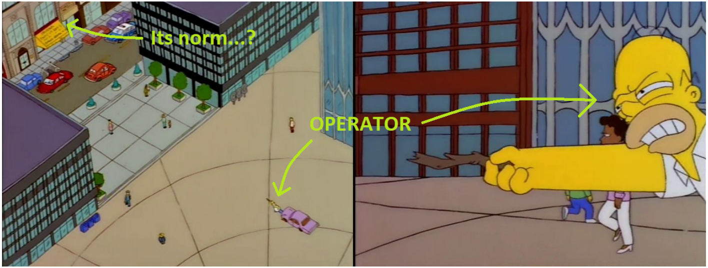

Figure: Operator trying (and failing) to attain its norm...?

Original pictures from The Simpsons, season 09, episode 01. Obtained at: https://frinkiac.com/caption/S09E01/712561

Norm-attainment. (Density of NA operators)

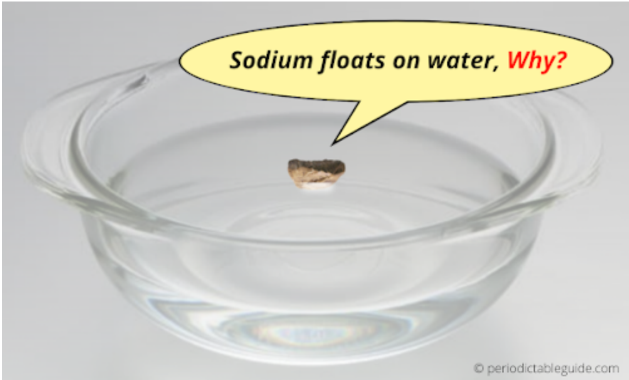

Figure: Na is not dense... (probably unrelated to the contents of the talk).

Original picture from:
https://periodictableguide.com/sodium-element-in-periodic-table/

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- Norm attaining operators.
- Norm attaining multilinear mappings.
- Norm attaining homogeneous polynomials.
- Norm attaining holomorphic functions.
- ...

Other norm attaining concepts

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$B \in \mathcal{B}(X \times Y, Z)$ attains its norm if there is $(x_0, y_0) \in S_X \times S_Y$ such that $\|B(x_0, y_0)\| = \|B\| = \sup_{(x,y) \in S_X \times S_Y} \|B(x, y)\|$.

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We will use $\| \cdot \|_\pi$ (resp. $\| \cdot \|_{\mathcal{N}}$) to approximate an element $z \in X \widehat{\otimes}_\pi Y$ (resp. $T \in \mathcal{N}(X, Y)$) by an element $z' \in \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$ (resp. $T' \in \mathrm{NA}_{\mathcal{N}}(X, Y)$). Even when not specified, density of norm attaining elements in these spaces will always be in terms of those norms during this talk.

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Note that $\mathcal{F}(X, Y) \subset \mathcal{N}(X, Y) \subset \mathcal{K}(X, Y)$, and that projective tensors are closely related to operators, bilinear mappings and nuclear operators.

Index

Tool for tensors

Theorem

Let X, Y be Banach spaces. Let $z \in X \widehat{\otimes}_\pi Y$ with

$$z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$. TFAE:

(1) $z \in \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$.

(2) $\exists G \in S_{\mathcal{L}(X,Y^*)}$ such that $G(x_n)(y_n) = 1, \forall n$.

(3) $\forall G \in S_{\mathcal{L}(X,Y^*)}$, $G(z) = \|z\|_\pi$ satisfies $G(x_n)(y_n) = 1, \forall n$.

Tool for nuclear operators

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Let X, Y be Banach spaces. Let $T \in \mathcal{N}(X, Y)$ with

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(3) $\forall G \in (\ker J)^{\perp}, \|G\| = 1, G(T) = \|T\|_{\mathcal{N}} \Longrightarrow G(x_n^*)(y_n) = 1, \forall n$.

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2) If H is a complex Hilbert space, then,

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Remark

Compare item 3) with this classical result:

If X is a Banach space such that $\mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every Banach space Y, then X must be reflexive.

Question 2

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QUESTION

Is it natural to ask whether or not the equalities

$$\mathrm{NA}_{\mathcal{N}}(X, Y) = \mathcal{N}(X, Y), \quad \text{and} \quad \mathrm{NA}_{\pi}(X \widehat{\otimes}_{\pi} Y) = X \widehat{\otimes}_{\pi} Y$$

hold for any arbitrary Banach spaces X and Y.

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Let X and Y be Banach spaces such that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$. Then

$$\overline{\mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})}^{\|\cdot\|} = \mathcal{B}(X \times Y, \mathbb{K}) \quad \left(\Longrightarrow \overline{\mathrm{NA}(X, Y^*)}^{\|\cdot\|} = \mathcal{L}(X, Y^*)\right).$$

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Negative examples

- $X = Y = L_1[0,1] \Rightarrow \mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$ not dense in $\mathcal{B}(X \times Y, \mathbb{K})$.
(Y. S. Choi, 1997).
- $X = L_1[0,1]$, Y^* strictly convex Banach space without the RNP $\Rightarrow$ $\mathrm{NA}(X, Y^*)$ not dense in $\mathcal{L}(X, Y^*)$.
(J. J. Uhl, 1976).
- There is a Banach space G such that $\mathrm{NA}_{\mathcal{B}}(G \times \ell_p, \mathbb{K})$ is not dense in $\mathcal{B}(G \times \ell_p, \mathbb{K})$ for $1 < p < \infty$.
(W. T. Gowers, 1990).

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Let X and Y be Banach spaces. If $B \in \mathcal{B}(X \times Y, \mathbb{K}) = (X \widehat{\otimes}_\pi Y)^*$ attains its norm as a functional at some $z \in \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$, then $B \in \mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$.

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Let X and Y be Banach spaces. If $B \in \mathcal{B}(X \times Y, \mathbb{K}) = (X \widehat{\otimes}_{\pi} Y)^*$ attains its norm as a functional at some $z \in \mathrm{NA}_{\pi}(X \widehat{\otimes}_{\pi} Y)$, then $B \in \mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$.

Negative example

- Let $X = L_1(\mathbb{T})$, where $\mathbb{T}$ is the unit circle equipped with the Haar measure m, and let Y be the 2-dimensional Hilbert space. Then, $\exists T \in \mathcal{B}(X \times Y, \mathbb{K})$ which attains its norm as a linear functional on $X \widehat{\otimes}_{\pi} Y$, but not as an operator from X into Y^* (nor the more as a bilinear form on $X \times Y$). (G. Godefroy, 2015).

Question 3

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But we have also seen many examples where not every tensor or nuclear operator attains its respective norm.

QUESTION

What can we say about the density of norm attaining tensors and nuclear operators? Do we always have density of such norm attaining elements?

Question 3

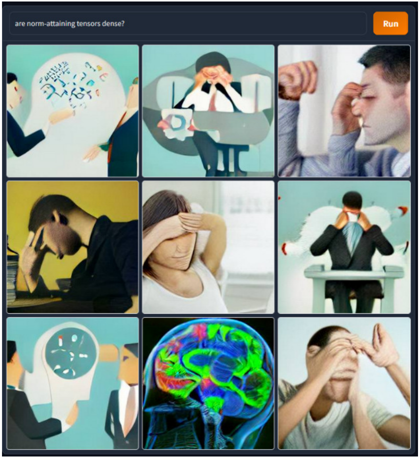

Figure: Asking an AI if norm-attaining tensors are dense.

Pictures generated by AI at https://huggingface.co/spaces/dalle-mini/dalle-mini

The $\mathbf{L}_{o,o}$ and the $\mathbf{L}_{o,o,\mathcal{B}}$

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Let X, Y and Z be Banach spaces.

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Let X, Y and Z be Banach spaces.

(a) (X, Y) has the $\mathbf{L}_{o,o}$ for operators if given $\varepsilon > 0$ and $T \in S_{\mathcal{L}(X,Y)}$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies $\|T(x)\| > 1 - \eta(\varepsilon, T)$, there is $x_0 \in S_X$ such that $\|T(x_0)\| = 1$ and $\|x_0 - x\| < \varepsilon$.

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(b) $(X \times Y, Z)$ satisfies the $\mathbf{L}_{o,o}$ for bilinear mappings ($\mathbf{L}_{o,o,\mathcal{B}}$) if given $\varepsilon > 0$ and $B \in \mathcal{B}(X \times Y, Z)$ with $\|B\| = 1$, there exists $\eta(\varepsilon, B) > 0$ such that whenever $(x, y) \in S_X \times S_Y$ satisfies $\|B(x, y)\| > 1 - \eta(\varepsilon, B)$, there is $(x_0, y_0) \in S_X \times S_Y$ such that $\|B(x_0, y_0)\| = 1$, $\|x - x_0\| < \varepsilon$, and $\|y - y_0\| < \varepsilon$.

Examples (Dantas-Kim-Lee-Mazzitelli, 2020)

(a) $\dim(X), \dim(Y) < \infty \implies (X \times Y, Z)$ has the $\mathbf{L}_{o,o,\mathcal{B}}$ $\forall Z$ Banach.

(b) If Y is unif. conv., $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o,\mathcal{B}}$ $\iff$ (X, Y^*) has the $\mathbf{L}_{o,o}$.

(c) If $1 < p, q < \infty$, then $(\ell_p \times \ell_q, \mathbb{K})$ has the $\mathbf{L}_{o,o,\mathcal{B}}$ if and only if $p > q'$.

(d) There are reflexive spaces X, Y s.t. $(X \times Y, \mathbb{K})$ fails the $\mathbf{L}_{o,o,\mathcal{B}}$.

First results on density

Theorem

Let X, Y be Banach spaces.

(a) If $(X^* \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o,\mathcal{B}}$, then

$$\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X,Y).$$

(b) If $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o,\mathcal{B}}$, then

$$\overline{\mathrm{NA}_{\pi}(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_{\pi}} = X\widehat{\otimes}_\pi Y.$$

Corollary

Let X be a finite-dimensional Banach space, and let Y be a Banach space. Then if Y is finite dimensional or uniformly convex, we have:

$$\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X,Y), \quad \overline{\mathrm{NA}_{\pi}(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_{\pi}} = X\widehat{\otimes}_\pi Y.$$

Metric π-property

Definition (metric π-property)

A Banach space X has **metric π-property** if given $\varepsilon > 0$ and $\{x_1, \dots, x_n\} \subseteq S_X$, there is a finite dimensional 1-complemented subspace $M \subseteq X$ and there are $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$, for every $i \in \{1, \dots, n\}$.

- This is equivalent to what is called metric π-property as an approximation property. Check P. G. Casazza's chapter on approximation properties.

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Spaces with metric π-property

(a) Banach spaces with monotone Schauder basis.

(b) $L_p(\mu)$-spaces for any $1 \leq p < \infty$ and any measure μ.

(c) Isometric predual spaces of L_1.

(d) If $\{X_n\}_{n\in\mathbb{N}}$ have the metric π-property, then so do $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$ and $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell_p}$, with $1 \leq p < \infty$.

(e) If X and Y have the metric π-property, then so do $X\widehat{\otimes}_\pi Y$ (projective tensor product), $X\widehat{\otimes}_\varepsilon Y$ (injective tensor product) and $X \oplus_a Y$ (absolute sum).

More results on density

Theorem

Let X be a Banach space satisfying metric π-property.

(a) If Y satisfies metric π-property, then $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$.

(b) If Y is uniformly convex, then $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$.

Remark

Metric π-property $\Longrightarrow$ metric approximation property $\Longrightarrow$ A.P.

Corollary

Let X be Banach space such that X^* satisfies metric π-property.

(a) If Y satisfies metric π-property, then $\overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X, Y)$.

(b) If Y is uniformly convex, then $\overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X, Y)$.

Density results from further research

S. Dantas, L. C. García-Lirola, M. Jung, A. Rueda Zoca, On norm-attainment in (symmetric) tensor products, *Quaestiones Mathematicae* (2022).

They got more density results involving dual spaces, approximation properties and Radon-Nikodym Property. For example:

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Theorem (Dantas, García-Lirola, Jung, Rueda Zoca)

a) Y dual $\Longrightarrow \overline{\mathrm{NA}_\pi(c_0 \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = c_0 \widehat{\otimes}_\pi Y$.

b) X^* and Y^* RNP and one has AP $\Longrightarrow \overline{\mathrm{NA}_\pi(X^* \widehat{\otimes}_\pi Y^*)}^{\|\cdot\|_\pi} = X^* \widehat{\otimes}_\pi Y^*$.

c) X and Y reflexive and one has AP $\Longrightarrow \overline{\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \widehat{\otimes}_\pi Y$.

Question 4

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It seems that many spaces satisfy the density of norm attaining tensors and nuclear operators.

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QUESTION

Are the following equalities true in general for all Banach spaces X and Y?

(a) $\overline{\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X \widehat{\otimes}_\pi Y.$

(b) $\overline{\mathrm{NA}_\mathcal{N}(X, Y)}^{\|\cdot\|_\mathcal{N}} = \mathcal{N}(X, Y).$

A negative result for tensors

A negative result for tensors

Theorem

A negative result for tensors

Theorem

Let $\mathcal{R}$ be Read's space. There exists a subspace X of c_0 without the approximation property and a subspace Y of $\mathcal{R}$ such that

$$\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)}^{\|\cdot\|_\pi} \neq X\widehat{\otimes}_\pi Y^*.$$

Read's space

Read's space is a renorming of c_0 satisfying deep properties that we will not use here. We will use, though, that its bidual is strictly convex.

A negative result for tensors

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Let $\mathcal{R}$ be Read's space. There exists a subspace X of c_0 without the approximation property and a subspace Y of $\mathcal{R}$ such that

$$\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)}^{\|\cdot\|_\pi} \neq X\widehat{\otimes}_\pi Y^*.$$

Idea: Recall that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)$ is linked with $\mathrm{NA}(X, Y^{**})$.

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Idea: Recall that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)$ is linked with $\mathrm{NA}(X, Y^{**})$. We work in a setting where:

- The lack of approximation property allows us to have operators that can't be approximated by finite rank operators.

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Theorem

Let $\mathcal{R}$ be Read's space. There exists a subspace X of c_0 without the approximation property and a subspace Y of $\mathcal{R}$ such that

$$\overline{\mathrm{NA}_{\pi}(X\widehat{\otimes}_{\pi}Y^{*})}^{\|\cdot\|_{\pi}} \neq X\widehat{\otimes}_{\pi}Y^{*}.$$

Idea: Recall that $\mathrm{NA}_{\pi}(X\widehat{\otimes}_{\pi}Y^{*})$ is linked with $\mathrm{NA}(X, Y^{**})$. We work in a setting where:

- The lack of approximation property allows us to have operators that can't be approximated by finite rank operators.
- The only norm attaining operators have finite rank.

A negative result for tensors

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Let $\mathcal{R}$ be Read's space. There exists a subspace X of c_0 without the approximation property and a subspace Y of $\mathcal{R}$ such that

$$\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)}^{\|\cdot\|_\pi} \neq X\widehat{\otimes}_\pi Y^*.$$

Idea: Recall that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)$ is linked with $\mathrm{NA}(X, Y^{**})$. We work in a setting where:

- The lack of approximation property allows us to have operators that can't be approximated by finite rank operators.
- The only norm attaining operators have finite rank.
- We prove that $\mathrm{NA}(X, Y^{**}) \cap B_{\mathcal{L}(X,Y^{**})}$ is not norming for $X\widehat{\otimes}_\pi Y^*$ here.

Further research

S. Dantas, L. C. García-Lirola, M. Jung, A. Rueda Zoca, On norm-attainment in (symmetric) tensor products, *Quaestiones Mathematicae* (2022).

The authors have studied similar questions involving:

- N-fold projective symmetric tensor product, $\widehat{\otimes}_{\pi,s,N} X$ (which is the completion of the space generated by the elements $\{x \otimes \overset{N}{\cdots} \otimes x : x \in X\}$ with the projective norm for those spaces).
- N-homogeneous polynomials $\mathcal{P}(^N X)$ (which is the dual of $\widehat{\otimes}_{\pi,s,N} X$).

Some more background for the interested reader

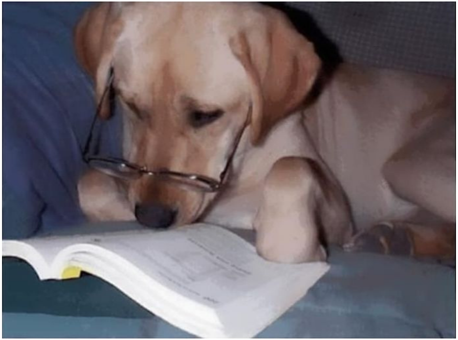

Figure: Interested reader (∗)

(∗): Picture from:
https://www.estandarte.com/noticias/varios/fotos-de-perros-leyendo_2352.html

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Thank you for your attention!

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