On the maximal extension in the mixed ultradifferentiable weight sequence setting

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Motivation - I

- (∗) Let us consider the so-called Borel map j^∞ (at 0) defined by $j^\infty(f) := (f^{(j)}(0))_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$.
- (∗) f is a given smooth function. In the smooth category (written $\mathcal{E}$), j^∞ is surjective but not injective (=existence of smooth flat function at 0).
- (∗) Restrict j^∞ to sub-classes of $\mathcal{E}$ and ask for surjectivity and injectivity (on "canonical target/sequence spaces")...
- (∗) We focus on so-called ultradifferentiable classes - derivatives of f are weighted w.r.t. sequences M, functions ω or matrices $\mathcal{M}$.
- (∗) In this talk: M (Roumieu- and Beurling-type - first part), then $\mathcal{M}$ (only Roumieu-type - second part).

Motivation - II

(∗) Known: Surjectivity and injectivity can be characterized in terms of $M=(M_j)_{j\in\mathbb{N}}$ (satisfying weak standard assumptions).

(∗) j^∞ is injective (Mandelbrojt, Hörmander, Komatsu) ("Denjoy-Carleman-theorem"), if and only M is quasianalytic, i.e.

$$\sum_{j=1}^{+\infty}\frac{M_{j-1}}{M_j}=+\infty.$$

(∗) j^∞ is surjective (Petzsche), if and only if M is strong non-quasianalytic ($(M.3)$, (γ_1)), i.e.

$$\sup_{j\in\mathbb{N}_{>0}}\frac{M_j/M_{j-1}}{j}\sum_{k\geq j}\frac{M_{k-1}}{M_k}<+\infty.$$

Motivation - III

- (∗) What happens when N is non-quasianalytic but not strongly non-quasianalytic?
- (∗) A "controlled loss" of regularity is possible (Schmets and Valdivia; Jiménez-Garrido, Sanz, and S.)

  Controlled loss: between N and a smaller M. - Different (!!) mixed conditions between M and N naturally appear.
- (∗) Question (J. Sanz): Which sequence M is optimal ("the largest one") when N is given and fixed?
- (∗) We try to answer this question now...

Sequences and conditions - I

- (∗) Let $M = (M_j)_j \in \mathbb{R}_{>0}^{\mathbb{N}}$ we also use $m_j := \frac{M_j}{j!}$ and $\mu_j := \frac{M_j}{M_{j-1}}$, $\mu_0 := 1$. W.l.o.g. $1 = M_0 \leq M_1$ (normalization).
- (∗) M is called *log-convex* ($(M.1)$) if

$$\forall\, j \in \mathbb{N}_{>0} : \; M_j^2 \leq M_{j-1} M_{j+1},$$

  equivalently if $(\mu_j)_j$ is nondecreasing.
- (∗) Consider the set

$$\mathcal{LC} := \{M \in \mathbb{R}_{>0}^{\mathbb{N}} : \quad \text{normalized, log-convex, } \lim_{j \to +\infty} (M_j)^{1/j} = +\infty\}.$$

- (∗) M has *moderate growth* ($(M.2)$) - write (mg)), if

$$\exists\, C \geq 1\; \forall\, j, k \in \mathbb{N} : \; M_{j+k} \leq C^{j+k} M_j M_k.$$

Sequences and conditions - II

(∗) Write $M \preceq N$ if

$$\sup_{j\in\mathbb{N}_{>0}}\left(\frac{M_j}{N_j}\right)^{1/j}<+\infty.$$

(∗) We call M and N *equivalent*, written $M \approx N$, if $M \preceq N$ and $N \preceq M$.

(∗) Finally, write $M \leq N$ if $M_j \leq N_j$ for all $j \in \mathbb{N}$.

Ultradifferentiable (test function) classes - I

(∗) Let $N \in \mathbb{R}_{>0}^{\mathbb{N}}$ and $h > 0$. Define the Banach space

$$\mathcal{D}_{N,h}([-1,1]) :=$$
$$\left\{ f \in \mathcal{E}(\mathbb{R},\mathbb{C}) : \operatorname{supp}(f) \subseteq [-1,1], \sup_{j\in\mathbb{N},x\in\mathbb{R}} \frac{|f^{(j)}(x)|}{h^j N_j} < +\infty \right\}.$$

(∗) The *ultradifferentiable test function class of Roumieu-type* is given by

$$\mathcal{D}_{\{N\}}([-1,1]) := \varinjlim_{h>0} \mathcal{D}_{N,h}([-1,1]),$$

a countable (LB)-space.

(∗) The *Beurling-type class* is given by

$$\mathcal{D}_{(N)}([-1,1]) := \varprojlim_{h>0} \mathcal{D}_{N,h}([-1,1]),$$

which is a Fréchet space.

Ultradifferentiable classes - II

(∗) Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and $h > 0$. For $\mathbf{a} := (a_j)_j \in \mathbb{C}^{\mathbb{N}}$ we put

$$|\mathbf{a}|_{M,h} := \sup_{j \in \mathbb{N}} \frac{|a_j|}{h^j M_j}, \qquad \Lambda_{M,h} := \{(a_j)_j \in \mathbb{C}^{\mathbb{N}} : |\mathbf{a}|_{M,h} < +\infty\}.$$

(∗) Furthermore we set

$$\Lambda_{(M)} := \{(a_p)_p \in \mathbb{C}^{\mathbb{N}} : \forall\, h > 0 : |\mathbf{a}|_{M,h} < +\infty\},$$

and

$$\Lambda_{\{M\}} := \{(a_p)_p \in \mathbb{C}^{\mathbb{N}} : \exists\, h > 0 : |\mathbf{a}|_{M,h} < +\infty\}.$$

(∗) We endow $\Lambda_{(M)}$ resp. $\Lambda_{\{M\}}$ with a natural projective, respectively inductive, topology via

$$\Lambda_{(M)} = \varprojlim_{h>0} \Lambda_{M,h}, \qquad \Lambda_{\{M\}} = \varinjlim_{h>0} \Lambda_{M,h}.$$

Ultradifferentiable classes - III

Write $[\cdot]$ if either $\{\cdot\}$ or $(\cdot)$.

If $M \preceq M'$, then

- $(*)$ $\Lambda_{[M]} \subseteq \Lambda_{[M']}$ and
- $(*)$ $\mathcal{D}_{[M]}([-1,1]) \subseteq \mathcal{D}_{[M']}([-1,1])$.

"Denjoy-Carleman-Theorem":

Theorem

Let $N \in \mathcal{LC}$, then $\mathcal{D}_{[N]}([-1,1]) \neq \{0\}$ if and only if N is non-quasianalytic, i.e.

$$\sum_{j=1}^{+\infty} \frac{1}{\nu_j} < +\infty.$$

Classes of sequences - I

Let $N \in \mathcal{LC}$, then introduce the sets

$$\mathcal{N}_{\preceq,R} := \{M \in \mathbb{R}_{>0}^{\mathbb{N}} : \quad \liminf_{j\to+\infty}(m_j)^{1/j} > 0, \quad M \preceq N\},$$

$$\mathcal{N}_{\preceq,B} := \{M \in \mathbb{R}_{>0}^{\mathbb{N}} : \quad \lim_{j\to+\infty}(m_j)^{1/j} = +\infty, \quad M \preceq N\}.$$

- $(*)$ $\mathcal{N}_{\preceq,B} \subseteq \mathcal{N}_{\preceq,R}$,
- $(*)$ if $\liminf_{p\to+\infty}(n_j)^{1/j} > 0$, then $N \in \mathcal{N}_{\preceq,R}$,
- $(*)$ if $\lim_{p\to+\infty}(n_j)^{1/j} = +\infty$, then $N \in \mathcal{N}_{\preceq,B}$.

Classes of sequences - II

We also introduce the smaller sets

$$\mathcal{N}_{\preceq,\mathcal{LC},R} := \{M \in \mathcal{N}_{\preceq,R} : \quad M \in \mathcal{LC}\},$$

and

$$\mathcal{N}_{\preceq,\mathcal{LC},B} := \{M \in \mathcal{N}_{\preceq,B} : \quad M \in \mathcal{LC}\}.$$

One can replace $M \preceq N$ by $M \leq CN$ ("replace $\preceq$ by $\leq$"):

$M \preceq N$ means $M_j \leq C^j N_j$; then replace M by the equivalent sequence $M^C := (M_j/C^j)_{j\in\mathbb{N}}$.

Relevant mixed conditions

(∗) Let $N \in \mathcal{LC}$ be non-quasianalytic and $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ ($M_0 = 1$).

(∗) Write $(M,N)_{\gamma_1}$ (Chaumat/Chollet '94) if

$$\sup_{j\in\mathbb{N}_{>0}} \frac{\mu_j}{j} \sum_{k\ge j} \frac{1}{\nu_k} < +\infty.$$

(∗) Obviously $(N,N)_{\gamma_1}$ is (γ_1) for N.

(∗) Write $(M,N)_{SV}$ (Schmets/Valdivia '03) if

$$\exists\ s \in \mathbb{N}_{>0} : \quad \sup_{p\in\mathbb{N}_{>0}} \frac{\lambda^{M,N}_{p,s}}{p} \sum_{k\ge p} \frac{1}{\nu_k} < +\infty,$$

with $\lambda^{M,N}_{p,s} := \sup_{0\le j<p} \left(\frac{M_p}{s^p N_j}\right)^{1/(p-j)}$.

Known comparison of relevant mixed conditions

Let $N \in \mathcal{LC}$ be non-quasianalytic.

(∗) If $M \in \mathcal{N}_{\le,\mathcal{LC},R}$ ($M \in \mathcal{N}_{\le,\mathcal{LC},B}$), then

$$\forall\ p,s \in \mathbb{N}_{>0}: \quad \lambda^{M,N}_{p,s} \le C \min\{\mu_p, \nu_p\},$$

when $M \le CN$. Thus

$$(M,N)_{\gamma_1} \Rightarrow (M,N)_{SV}.$$

(∗) If M has in addition (mg), equivalently $\sup_{j\in\mathbb{N}_{>0}} \frac{\mu_j}{(M_j)^{1/j}} < +\infty$ (!!!), then

$$(M,N)_{\gamma_1} \Leftrightarrow (M,N)_{SV}.$$

(∗) $(M,N)_{\gamma_1}$ resp. $(M,N)_{SV}$ is valid if and only if $(M^C,N)_{\gamma_1}$ resp. $(M^C,N)_{SV}$.

Main (known) characterizing result

By Schmets/Valdivia (and Jiménez-Garrido, Sanz, S. - formulated for $M \in \mathcal{N}_{\preceq,\mathcal{LC},R}$ resp. $M \in \mathcal{N}_{\preceq,\mathcal{LC},B}$) we know:

Theorem

Let $N \in \mathcal{LC}$ and $M \in \mathcal{N}_{\preceq,R}$ ($M \in \mathcal{N}_{\preceq,B}$). Then the following are equivalent:

(i) $j^\infty(\mathcal{D}_{\{N\}}([-1,1])) \supseteq \Lambda_{\{M\}}$ *resp.* $j^\infty(\mathcal{D}_{(N)}([-1,1])) \supseteq \Lambda_{(M)}$,

(ii) $(M,N)_{SV}$ *is valid.*

Also known: (γ_1) for N is $(N,N)_{\gamma_1} \Leftrightarrow (N,N)_{SV}$.

Question: Optimal sequences for the mixed conditions and comparison between them?

(Known) optimal sequence for $(\cdot, N)_{\gamma_1}$

Let $N \in \mathcal{LC}$ be non-quasianalytic.
Let the descendant S^N be defined by its quotients

$$\sigma_p^N := \frac{\tau_1 p}{\tau_p}, \quad p \in \mathbb{N}_{>0}, \qquad \sigma_0^N := 1,$$

with

$$\tau_p := \frac{p}{\nu_p} + \sum_{j \geq p} \frac{1}{\nu_j}, \quad p \geq 1.$$

Properties:

(i) $s^N := (S_p^N/p!)_{p\in\mathbb{N}} \in \mathcal{LC}$ (i.e. S^N is strongly log-convex),

(ii) $\exists\, C > 0\ \forall\, p \in \mathbb{N}: \quad \sigma_p^N \leq C\nu_p$,

(iii) $(S^N, N)_{\gamma_1}$,

(iv) if N has (mg), then S^N, too.

Optimality of the descendant

Summarizing:

(∗) $S^N \in \mathcal{N}_{\preceq,\mathcal{LC},B} \subseteq \mathcal{N}_{\preceq,\mathcal{LC},R}$,

(∗) $(S^N, N)_{SV}$,

(∗) if $M \in \mathcal{LC}$ with $\mu_p \leq C\nu_p$ and $(M,N)_{\gamma_1}$, then $\mu_p \leq D\sigma_p^N$ follows for some $D \geq 1$ (implies $\preceq$-maximality),

(∗) $S^N \approx N$ if and only if N satisfies (γ_1) (if and only if $(N,N)_{SV}$).

Definition of S^N is inspired by the non-mixed approach in Petzsche; has been given by Rainer/S.

Optimal sequence for relation $(\cdot, N)_{SV}$

Goal: Determine the $\preceq$-maximal sequence L belonging to $\mathcal{N}_{\preceq,R}$ resp. $\mathcal{N}_{\preceq,B}$ and satisfying $(L,N)_{SV}$.

Let $N \in \mathcal{LC}$ and non-quasianalytic. Let $s \in \mathbb{N}_{>0}$ and define $L^s = (L^s_p)_{p\in\mathbb{N}}$ by

$$L^s_0 := 1, \qquad L^s_p := s^p \min_{0\le j\le p-1}\left\{\left(\frac{p}{\sum_{k\ge p}\frac{1}{\nu_k}}\right)^{p-j} N_j\right\}, \quad p \in \mathbb{N}_{>0}.$$

Theorem

Let $N \in \mathcal{LC}$ and non-quasianalytic. Then:

- (*i*) $(L^s, N)_{SV}$ *holds true for all* $s \in \mathbb{N}_{>0}$ *(note that* $L^s \approx L^t$ *for all* $s, t \in \mathbb{N}_{>0}$*),*
- (*ii*) $M \preceq L^s$ *for any* $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ *satisfying* $(M, N)_{SV}$ *and for all* $s \in \mathbb{N}_{>0}$*,*
- (*iii*) $L^s \preceq N$ *for all* $s \in \mathbb{N}_{>0}$*.*

- ($*$) There exists $M \in \mathcal{N}_{\preceq,\mathcal{LC},R}$ ($M \in \mathcal{N}_{\preceq,\mathcal{LC},B}$) with $(M, N)_{SV}$ - take $M \equiv S^N$!
- ($*$) (i), (ii) and (iii) hold true for the log-convex minorant $\underline{L^s}$ as well (with (ii) restricted to all log-convex sequences M).

(∗) Let $M \in \mathcal{N}_{\preceq,R}$ ($M \in \mathcal{N}_{\preceq,B}$) with $(M,N)_{SV}$, then $M \preceq L^s$ for some/each $s \in \mathbb{N}_{>0}$.

(∗) $L^s \in \mathcal{N}_{\preceq,B} \subseteq \mathcal{N}_{\preceq,R}$.
Consequently, some/each L^s is $\preceq$-maximal among all $M \in \mathcal{N}_{\preceq,B}$ ($M \in \mathcal{N}_{\preceq,R}$) and having $(M,N)_{SV}$.

(∗) The same holds true for $\underline{L^s}$ instead of L^s.

(∗) Formally $\underline{L^s} \in \mathcal{N}_{\preceq,\mathcal{LC},B}$ is not clear since normalization might fail.
$\underline{L^s}$ is equivalent to $\underline{\widetilde{L^s}} \in \mathcal{N}_{\preceq,\mathcal{LC},B}$; $\underline{\widetilde{L^s}}$ is $\preceq$-maximal among all $M \in \mathcal{N}_{\preceq,\mathcal{LC},B}$ ($M \in \mathcal{N}_{\preceq,\mathcal{LC},R}$).

Main characterizing result - reformulated

Theorem

Let $N \in \mathcal{LC}$. Then the set of sequences $M \in \mathcal{N}_{\preceq,R}$ ($M \in \mathcal{N}_{\preceq,B}$) and satisfying

$$j^{\infty}(\mathcal{D}_{\{N\}}([-1,1])) \supseteq \Lambda_{\{M\}}, \quad \text{resp.} \quad j^{\infty}(\mathcal{D}_{(N)}([-1,1])) \supseteq \Lambda_{(M)},$$

has a maximal element which is given by some/each L^s.

Alternatively, we can use $\underline{\widetilde{L^s}}$ which is $\preceq$-maximal among all $M \in \mathcal{N}_{\preceq,\mathcal{LC},R}$ ($M \in \mathcal{N}_{\preceq,\mathcal{LC},B}$).

Note: non-q.a. is not required as an assumption; it is a necessary condition.

Testing (γ_1) with optimal sequence

Lemma

Let $N \in \mathcal{LC}$.

(*i*) *If N satisfies (γ_1), i.e. N is strong non-quasianalytic, then $N \approx L^s$ for all $s \in \mathbb{N}_{>0}$.*

(*ii*) *Conversely, if $N \approx L^s$ for some/each $s \in \mathbb{N}_{>0}$, then N satisfies (γ_1).*

Comparison of S^N and L^s - I

- $(*)$ Automatically we have $S^N \preceq L^s$ (for some/each $s \in \mathbb{N}_{>0}$).
- $(*)$ What about the converse?
- $(*)$ Advantage of S^N: "Easier" to compute and better regularity properties.

Introduce

$$\liminf_{p\to+\infty} \frac{\nu_p}{p} \sum_{k\geq 2p} \frac{1}{\nu_k} > 0. \tag{1}$$

Lemma

Let $N \in \mathcal{LC}$ be non-quasianalytic. If N satisfies (mg), *then* (1) *holds true, but the converse fails in general.*

Importance: (1) for N implies (mg) for $S^N \Rightarrow$ comparison between σ_j^N and $(S_j^N)^{1/j}$ is possible!

Comparison of S^N and L^s - II

Theorem

Let $N \in \mathcal{LC}$ satisfy

$$\liminf_{p\to+\infty} \frac{\nu_p}{p} \sum_{k\ge 2p} \frac{1}{\nu_k} > 0.$$

Then for all $s \in \mathbb{N}_{>0}$ we have $S^N \approx L^s$.

Thus S^N is $\preceq$-maximal among all $M \in \mathcal{N}_{\preceq,R}$ ($M \in \mathcal{N}_{\preceq,B}$) and also among all $M \in \mathcal{N}_{\preceq,\mathcal{LC},R}$ ($M \in \mathcal{N}_{\preceq,\mathcal{LC},B}$) satisfying $(M,N)_{SV}$.

Equivalently, the inclusions $j^\infty(\mathcal{D}_{[N]}([-1,1])) \supseteq \Lambda_{[S^N]}$ are optimal in the ultradifferentiable setting.

A difference can occur

In general $S^N \approx L^s$ fails:

Lemma

There exist non-quasianalytic $N \in \mathcal{LC}$ such that $S^N \approx L^s$ cannot hold true.
In this case (γ_1) for N is violated.

- (∗) The reason of failure: $p \mapsto \frac{\nu_p}{p} \sum_{k \geq p} \frac{1}{\nu_k}$ can behave very irregular/oscillating.
- (∗) The construction of N: "quite technical"....
- (∗) S^N is defined in terms of quotients σ^N; L^s is given directly.
- (∗) A comparison of sequences of roots and quotients is convenient; (mg) (for S^N) guarantees this.
- (∗) Problem in the general matrix setting!

One more mixed condition - I

(∗) Carleson (R.-case) and Ehrenpreis (B.-case) have already studied these questions.... in terms of another mixed condition for associated weight functions.

(∗) Let $M \in \mathcal{LC}$, define the *associated weight function*

$$\omega_M(t) := \sup_{p\in\mathbb{N}} \log\left(\frac{t^p}{M_p}\right) \quad \text{for } t>0, \qquad \omega_M(0) := 0.$$

(∗) Consider relation $(\omega_M, \omega_N)_{\mathrm{snq}}$:

$$\exists C>0\ \forall t\geq 0: \int_1^{+\infty} \frac{\omega_N(ty)}{y^2}dy \leq C\omega_M(t)+C. \qquad (2)$$

(∗) Carleson (solving a moment problem) only treats the sufficiency of (2); Ehrenpreis shows also the necessity.

One more mixed condition - II

- $(*)$ (2) is crucial in the weight function setting (cf. Bonet/Meise/Taylor and Rainer/S.). Given ω (non-quasianalytic), then the heir

$$\kappa_\omega(t) := \int_1^\infty \frac{\omega(ts)}{s^2} ds = t \int_t^\infty \frac{\omega(s)}{s^2} ds$$

  is optimal.
- $(*)$ However, in Ehrenpreis an assumption is made on ω_N (on $\lambda_N \equiv \exp \circ \omega_N$) - precisely corresponding to (mg) for N - not needed for results above!
- $(*)$ But (mg) seems to be indispensable when involving weight function techniques to prove results for the weight sequence setting (cf. Bonet/Meise/Melikhov and Rainer/S.).
- $(*)$ Note: In $(\omega_M, \omega_N)_{\text{snq}}$ the constant C is "outside" the function!!

One more mixed condition - III

(∗) How is $(\omega_M, \omega_N)_{\mathrm{snq}}$ related to $(M,N)_{\gamma_1}$, $(M,N)_{SV}$? How about optimality? Recall: If $M \in \mathcal{LC}$, then

$$M_j = \sup_{t \geq 0} \frac{t^j}{\exp(\omega_M(t))}, \quad j \in \mathbb{N}.$$

(∗) Jiménez-Garrido/Rainer/Sanz/S. (Chaumat/Chollet):
If $M, N \in \mathcal{LC}$ with $\mu \leq \nu$ then $(M,N)_{\gamma_1} \Rightarrow (\omega_M, \omega_N)_{\mathrm{snq}}$ and $(\omega_M, \omega_N)_{\mathrm{snq}} \Rightarrow (M,N)_{\gamma_1}$ if M has in addition (mg).

(∗) Summarizing:
If $M, N \in \mathcal{LC}$ with $\mu \leq \nu$ and M has (mg), then

$$(\omega_M, \omega_N)_{\mathrm{snq}} \Leftrightarrow (M,N)_{\gamma_1} \Leftrightarrow (M,N)_{SV}.$$

Weight functions - I

- (∗) Let $\omega : [0, \infty) \to [0, \infty)$ be continuous increasing, $\omega(0) = 0$ and $\lim_{t\to\infty} \omega(t) = \infty$.
- (∗) ω is called *normalized* if $\omega(t) = 0$ for $t \in [0, 1]$.
- (∗) We call ω a *pre-weight function* if additionally
  - (∗) $\log(t) = o(\omega(t))$ as $t \to \infty$,
  - (∗) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex.
- (∗) A pre-weight function ω is a *weight function* if it also fulfills
  - (∗) $\omega(2t) = O(\omega(t))$ as $t \to \infty$.
- (∗) A pre-weight function ω is called *non-quasianalytic* if

$$\int_0^\infty \frac{\omega(t)}{1 + t^2} dt < \infty.$$

- (∗) A pre-weight function ω is called *strong (non-q.a.)* if $(\omega, \omega)_{\mathrm{snq}}$.

Weight functions - II

Let $M \in \mathcal{LC}$, then

- (∗) ω_M is a normalized pre-weight function,
- (∗) but in general not a weight function (very recently characterized by S.),
- (∗) M is non-quasianalytic if and only if ω_M is so.

Weight matrices

- $\mathcal{M} := \{M^{(x)} : x > 0\}$ is a weight matrix if
  - $\forall\, x > 0 : \quad M^{(x)} \in \mathcal{LC}$ and
  - $M^{(x)} \leq M^{(y)}$ if $x \leq y$.
- $\mathcal{M}$ is called non-quasianalytic if all $M^{(x)} \in \mathcal{M}$ are non-q.a.
- Write $\mathcal{M}\{\preceq\}\mathcal{N}$ if

$$\forall\, M^{(x)} \in \mathcal{M}\ \exists\, N^{(y)} \in \mathcal{N} : \quad M^{(x)} \preceq N^{(y)}.$$

- $\mathcal{M}$ and $\mathcal{N}$ are R-equivalent if $\mathcal{M}\{\preceq\}\mathcal{N}$ and $\mathcal{N}\{\preceq\}\mathcal{M}$.
- $\mathcal{M}$ is said to have R-moderate growth if

$$\forall\, M^{(x)}\ \exists\, M^{(y)}\ \exists\, C \geq 1\ \forall\, j,k \in \mathbb{N} : \quad M^{(x)}_{j+k} \leq C^{j+k} M^{(y)}_j M^{(y)}_k.$$

Ultradifferentiable weight matrix classes (of R.-type)

(∗) Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$. For $h > 0$ and $n \in \mathbb{N}_{>0}$ we define the Banach space

$$\mathcal{E}_{M,h}([-n,n]) := \left\{ f \in \mathcal{E}([-n,n]) : \sup_{t \in [-n,n],\, k \in \mathbb{N}} \frac{|f^{(k)}(t)|}{h^k M_k} < \infty \right\}.$$

(∗) Then the *Denjoy–Carleman classes of Roumieu type* is given by

$$\mathcal{E}_{\{M\}}(\mathbb{R}) := \varprojlim_{n \in \mathbb{N}_{>0}} \varinjlim_{h \in \mathbb{N}_{>0}} \mathcal{E}_{M,h}([-n,n]).$$

(∗) Finally, let $\mathcal{M} = \{M^{(x)} : x > 0\}$ and set

$$\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}) := \varprojlim_{n \in \mathbb{N}_{>0}} \varinjlim_{h \in \mathbb{N}_{>0}} \varinjlim_{M^{(x)} \in \mathcal{M}} \mathcal{E}_{M^{(x)},h}([-n,n]).$$

(∗) Classes $\Lambda_{\{\mathcal{M}\}}$ are defined analogously.

Ultradifferentiable weight function classes (of R.-type)

(∗) For any (normalized) pre-weight function ω consider the *Young conjugate* of φ_ω,

$$\varphi^*_\omega(s) := \sup\{st - \varphi_\omega(t) : t \geq 0\}, \quad s \geq 0.$$

(∗) Let ω be a normalized pre-weight function. For $h > 0$ and $n \in \mathbb{N}_{>0}$ we define

$$\mathcal{E}_{\omega,h}([-n,n]) := \left\{ f \in \mathcal{E}([-n,n]) : \sup_{t \in [-n,n],\, k \in \mathbb{N}} \frac{|f^{(k)}(t)|}{\exp(\frac{1}{h}\varphi^*_\omega(hk))} < \infty \right\}.$$

(∗) Then the *Braun–Meise–Taylor class of Roumieu type* is given by

$$\mathcal{E}_{\{\omega\}}(\mathbb{R}) := \varprojlim_{n \in \mathbb{N}_{>0}} \varinjlim_{h \in \mathbb{N}_{>0}} \mathcal{E}_{\omega,h}([-n,n]).$$

(∗) Classes $\Lambda_{\{\omega\}}$ are defined analogously.

Associated weight matrix - I

Let ω be a normalized pre-weight function, define $\mathcal{M}_\omega := \{W^{(x)} : x > 0\}$ by

$$W_j^{(x)} := \exp\left(\frac{1}{x}\varphi_\omega^*(xj)\right).$$

(∗) $\mathcal{M}_\omega$ always has R-moderate growth and

(∗) $\mathcal{M}_\omega$ is non-q.a. if and only if ω is non-q.a.

Associated weight matrix - II

By Rainer/S. we know:

Theorem

Let ω be a weight function. Then (as locally convex vector spaces)

$$\mathcal{E}_{\{\omega\}}(\mathbb{R}) = \mathcal{E}_{\{\mathcal{M}_\omega\}}(\mathbb{R}),$$

and similarly if $\mathcal{E} \leftrightarrow \Lambda$ (resp. if using other symbols/functors). Moreover, this holds for B.-type classes as well.

- $(\ast)$ $\mathcal{M}\{\preceq\}\mathcal{N}$ implies $\Lambda_{\{\mathcal{M}\}} \subseteq \Lambda_{\{\mathcal{N}\}}$ and $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}) \subseteq \mathcal{E}_{\{\mathcal{N}\}}(\mathbb{R})$.
- $(\ast)$ Thus R-equivalent matrices give the same classes.

More relevant (new) derived sequences/matrices - I

(∗) Let ω be a non-q.a. pre-weight function (set $\omega(t) := \omega(|t|)$ for $t \in \mathbb{R}$).

(∗) Consider the harmonic extension

$$P_\omega(x+iy) := \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{\omega(t)}{(x-t)^2+y^2}dt, \quad y \neq 0,$$

$$P_\omega(x+iy) := \omega(x), \quad y = 0.$$

(∗) We have

$$P_\omega(ir) \le \frac{4}{\pi}\kappa_\omega(r) \le 4P_\omega(ir), \quad r > 0.$$

(∗) We define

$$Q_k := \sup_{r>0}\frac{r^{k+\frac{1}{2}}}{\exp(\frac{1}{2}P_\omega(ir))}, \quad k \in \mathbb{N}.$$

More relevant (new) derived sequences/matrices - II

(∗) Let $M \in \mathcal{LC}$ be non-q.a. and set

$$\widetilde{\omega}_M(t) := \omega_M(t) + \log(1+t^2).$$

(∗) Let $\mathcal{M} = \{M^{(x)} : x > 0\}$ be a non-q.a. weight matrix, put

$$\kappa_{(x)} := \kappa_{\widetilde{\omega}_{M^{(x)}}}, \quad K_j^{(x)} := \exp(\varphi^*_{\kappa_{(x)}}(j)).$$

(∗) Moreover, consider

$$P_{(x)} := P_{\widetilde{\omega}_{M^{(x)}}}.$$

(∗) Then introduce the matrices

$$\mathcal{K} := \{K^{(x)} : x > 0\},$$

$$\mathcal{Q} := \{Q^{(x)} : x > 0\}.$$

Relation between $\mathcal{K}$ and $\mathcal{Q}$

Theorem

Let $\mathcal{M} := \{M^{(x)} : x > 0\}$ be a non-q.a. weight matrix. Then

- $(*)$ *$\mathcal{K}$ is a weight matrix such that $(K_j^{(x)}/j!)^{1/j} \to \infty$ and $K_j^{(x)}/M_j^{(x)}$ is bounded for all $x > 0$ and $j \in \mathbb{N}$.*
- $(*)$ *If $\mathcal{M}$ has R-moderate growth, then $\mathcal{K}$, too.*
- $(*)$ *In this case $\mathcal{K}$ and $\mathcal{Q}$ are R-equivalent.*

Carleson's result for matrices

Theorem

Let $\mathcal{M}$ be a non-quasianalytic weight matrix of R-moderate growth. Then

$$\Lambda_{\{\mathcal{K}\}} = \Lambda_{\{\mathcal{Q}\}} \subseteq j^{\infty}(\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R})).$$

Proof:

(∗) Direct generalization of Carleson's arguments to the weight matrix setting (solving a moment problem).

(∗) Techniques involve matrix $\mathcal{Q}$.

Schmets/Valdivia's result for matrices

Theorem

Let $\mathcal{M}$ be a non-quasianalytic weight matrix and $\mathcal{M}'$ be a one-parameter family of positive sequences such that $\liminf_{k\to\infty}(M'_j/j!)^{1/j} > 0$ for all $M' \in \mathcal{M}'$. Then the following are equivalent:

(*i*) $\Lambda_{\{\mathcal{M}'\}} \subseteq j^{\infty}(\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}))$.

(*ii*) *The SV-condition of R.-type holds true, i.e.*

$$\forall\, M' \in \mathcal{M}'\ \exists\, M \in \mathcal{M} : \quad (M', M)_{SV}.$$

Proof: Follows by the definition of matrix classes and the characterization of SV-condition in the weight sequence setting.

Remark: The B.-type is NOT so easy to handle - see next talk.

Consequences and more (expected) derived matrices

Corollary

Let $\mathcal{M}$ be a non-quasianalytic weight matrix of R-moderate growth. Then

$$\forall\, x > 0\, \exists\, y > 0 : \quad (K^{(x)}, M^{(y)})_{SV}.$$

Now let $\mathcal{M} = \{M^{(x)} : x > 0\}$ be non-q.a. and consider for all x (set parameter $s := 1$ and $L := L^1$):

- $(*)$ the descendant $S^{(x)}$, yielding the matrix $\mathcal{S}$;
- $(*)$ the sequence $L^{(x)}$, yielding the matrix $\mathcal{L}$;
- $(*)$ the log-convex minorant $\underline{L}^{(x)}$, yielding the matrix $\underline{\mathcal{L}}$.

Relations between derived matrices

From the weight sequence setting we know:

$$\forall\, x > 0: \quad S^{(x)} \preceq \underline{L}^{(x)} \leq L^{(x)}.$$

Theorem

Let $\mathcal{M} = \{M^{(x)} : x > 0\}$ be a non-quasianalytic weight matrix of R-moderate growth. Then

$$\mathcal{S}\{\preceq\}\mathcal{K}\{\preceq\}\underline{\mathcal{L}}\{\preceq\}\mathcal{L}.$$

Recall that in this case $\mathcal{K}$ and $\mathcal{Q}$ are R-equivalent.

Classical case - the weight function setting - I

Theorem

Let ω be a non-quasianalytic weight function. Then

$$\Lambda_{\{\kappa_\omega\}} = \Lambda_{\{\mathcal{K}\}} = \Lambda_{\{\mathcal{Q}\}} = \Lambda_{\{\underline{\mathcal{L}}\}},$$

and the families $\mathcal{K}$, $\mathcal{Q}$ and $\mathcal{L}$ are derived from $\mathcal{M}_\omega$.

Two things to show:

(*i*) The first equality: weight matrix techniques are required (R-moderate growth used!)

(*ii*) Prove that $\mathcal{K}$ and $\underline{\mathcal{L}}$ are R-equivalent - technical...

Classical case - the weight function setting - II

Combining

- (∗) the previous result and
- (∗) the characterization in terms of mixed SV-condition (of R.-type) and
- (∗) the weight matrix representations for ω-ultradiff. classes

we get that

$$\Lambda_{\{\kappa_\omega\}} \subseteq j^\infty(\mathcal{E}_{\{\omega\}}(\mathbb{R}))$$

is optimal!

This gives back the characterization by Bonet/Meise/Taylor using completely different techniques (Functional Analysis)!

ω is strong (non-q.a.), if and only if $\mathcal{K}$, $\mathcal{Q}$, $\underline{\mathcal{L}}$ and $\mathcal{L}$ are all R-equivalent to $\mathcal{M}_\omega$.

Classical case - the weight sequence setting

Theorem

Let $M \in \mathcal{LC}$ be non-quasianalytic and of moderate growth. Then the derived sequences S, K, Q and L are equivalent.

For the first part see:

G. Schindl, On the maximal extension in the mixed ultradifferentiable weight sequence setting, Studia Math. 263, no. 2, 209-240, 2022, DOI: 10.4064/sm200930-17-3.

For the second part see:

A. Rainer, D.N. Nenning, and G. Schindl, On optimal solutions of the Borel problem in the Roumieu case, 2021, submitted, available online at `https://arxiv.org/pdf/2112.08463.pdf`.