Factorization theorems in Denjoy-Carleman classes associated to representations of $(\mathbb{R}^d, +)$

Jasson Vindas

jasson.vindas@UGent.be

Ghent University

Workshop on Functional and Complex Analysis WFCA22
Valladolid, June 22, 2022

Introduction

Factorization theorems in modules over function algebras is an important subject with a long tradition in mathematical analysis.

A module $\mathcal{M}$ over a non-unital algebra $\mathcal{A}$ is said to have the strong factorization property if

$$\mathcal{M} = \mathcal{A} \cdot \mathcal{M} = \{a \cdot m \mid a \in \mathcal{A}, m \in \mathcal{M}\}.$$

It is said to have the weak factorization property if

$$\mathcal{M} = \operatorname{span}(\mathcal{A} \cdot \mathcal{M}).$$

We will present some new results about strong factorization:

1. A strong factorization theorem of Dixmier-Malliavin type for ultradifferentiable vectors of representations of $(\mathbb{R}^d, +)$.
2. We have established the strong factorization property for many families of convolution modules of ultradifferentiable functions. We will give some examples.

The talk is based on collaborative work with Andreas Debrouwere and Bojan Prangoski.

Factorization theorems in modules over function algebras is an important subject with a long tradition in mathematical analysis.

A module $\mathcal{M}$ over a non-unital algebra $\mathcal{A}$ is said to have the strong factorization property if

$$\mathcal{M} = \mathcal{A} \cdot \mathcal{M} = \{a \cdot m \mid a \in \mathcal{A}, m \in \mathcal{M}\}.$$

It is said to have the weak factorization property if

$$\mathcal{M} = \operatorname{span}(\mathcal{A} \cdot \mathcal{M}).$$

We will present some new results about strong factorization:

1. A strong factorization theorem of Dixmier-Malliavin type for ultradifferentiable vectors of representations of $(\mathbb{R}^d, +)$.
2. We have established the strong factorization property for many families of convolution modules of ultradifferentiable functions. We will give some examples.

The talk is based on collaborative work with Andreas Debrouwere and Bojan Prangoski.

Introduction

Factorization theorems in modules over function algebras is an important subject with a long tradition in mathematical analysis.

A module $\mathcal{M}$ over a non-unital algebra $\mathcal{A}$ is said to have the strong factorization property if

$$\mathcal{M} = \mathcal{A} \cdot \mathcal{M} = \{a \cdot m \,|\, a \in \mathcal{A}, m \in \mathcal{M}\}.$$

It is said to have the weak factorization property if

$$\mathcal{M} = \operatorname{span}(\mathcal{A} \cdot \mathcal{M}).$$

We will present some new results about strong factorization:

1. A strong factorization theorem of Dixmier-Malliavin type for ultradifferentiable vectors of representations of $(\mathbb{R}^d, +)$.
2. We have established the strong factorization property for many families of convolution modules of ultradifferentiable functions. We will give some examples.

The talk is based on collaborative work with Andreas Debrouwere and Bojan Prangoski.

Introduction

Factorization theorems in modules over function algebras is an important subject with a long tradition in mathematical analysis.

A module $\mathcal{M}$ over a non-unital algebra $\mathcal{A}$ is said to have the strong factorization property if

$$\mathcal{M} = \mathcal{A} \cdot \mathcal{M} = \{a \cdot m \mid a \in \mathcal{A}, m \in \mathcal{M}\}.$$

It is said to have the weak factorization property if

$$\mathcal{M} = \operatorname{span}(\mathcal{A} \cdot \mathcal{M}).$$

We will present some new results about strong factorization:

1. A strong factorization theorem of Dixmier-Malliavin type for ultradifferentiable vectors of representations of $(\mathbb{R}^d, +)$.
2. We have established the strong factorization property for many families of convolution modules of ultradifferentiable functions. We will give some examples.

The talk is based on collaborative work with Andreas Debrouwere and Bojan Prangoski.

Factorization in classical function algebras

- Factorization theorems on $\mathbb{T}$ go back to Salem and Zygmund.
- Rudin showed (1957-1958): $L^1(\mathbb{R}^d) = L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d)$.
- Cohen (1959) extended this result to the function algebra of a locally compact group G,

$$L^1(G) = L^1(G) * L^1(G).$$

- Hewitt (1964) used Cohen technique to prove a general factorization theorem for Banach modules.
- Cohen-Hewitt type factorization theorems also hold for various Fréchet modules.
- Essential hypothesis: existence of bounded approximative units on the algebra under consideration.
- Many locally convex algebras do not have bounded approximative units. Examples: many basic algebras of smooth functions occurring in analysis.

Factorization in classical function algebras

- Factorization theorems on $\mathbb{T}$ go back to Salem and Zygmund.
- Rudin showed (1957-1958): $L^1(\mathbb{R}^d) = L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d)$.
- Cohen (1959) extended this result to the function algebra of a locally compact group G,

$$L^1(G) = L^1(G) * L^1(G).$$

- Hewitt (1964) used Cohen technique to prove a general factorization theorem for Banach modules.
- Cohen-Hewitt type factorization theorems also hold for various Fréchet modules.
- Essential hypothesis: existence of bounded approximative units on the algebra under consideration.
- Many locally convex algebras do not have bounded approximative units. Examples: many basic algebras of smooth functions occurring in analysis.

Factorization in classical function algebras

- Factorization theorems on $\mathbb{T}$ go back to Salem and Zygmund.
- Rudin showed (1957-1958): $L^1(\mathbb{R}^d) = L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d)$.
- Cohen (1959) extended this result to the function algebra of a locally compact group G,

$$L^1(G) = L^1(G) * L^1(G).$$

- Hewitt (1964) used Cohen technique to prove a general factorization theorem for Banach modules.
- Cohen-Hewitt type factorization theorems also hold for various Fréchet modules.
- Essential hypothesis: existence of bounded approximative units on the algebra under consideration.
- Many locally convex algebras do not have bounded approximative units. Examples: many basic algebras of smooth functions occurring in analysis.

Factorization in classical function algebras

- Factorization theorems on $\mathbb{T}$ go back to Salem and Zygmund.
- Rudin showed (1957-1958): $L^1(\mathbb{R}^d) = L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d)$.
- Cohen (1959) extended this result to the function algebra of a locally compact group G,

$$L^1(G) = L^1(G) * L^1(G).$$

- Hewitt (1964) used Cohen technique to prove a general factorization theorem for Banach modules.
- Cohen-Hewitt type factorization theorems also hold for various Fréchet modules.
- Essential hypothesis: existence of bounded approximative units on the algebra under consideration.
- Many locally convex algebras do not have bounded approximative units. Examples: many basic algebras of smooth functions occurring in analysis.

Factorization in classical function algebras

- Factorization theorems on $\mathbb{T}$ go back to Salem and Zygmund.
- Rudin showed (1957-1958): $L^1(\mathbb{R}^d) = L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d)$.
- Cohen (1959) extended this result to the function algebra of a locally compact group G,

$$L^1(G) = L^1(G) * L^1(G).$$

- Hewitt (1964) used Cohen technique to prove a general factorization theorem for Banach modules.
- Cohen-Hewitt type factorization theorems also hold for various Fréchet modules.
- Essential hypothesis: existence of bounded approximative units on the algebra under consideration.
- Many locally convex algebras do not have bounded approximative units. Examples: many basic algebras of smooth functions occurring in analysis.

Factorization in algebras of smooth functions

- Ehrenpreis' problem (1960):

Does $\mathcal{D}(\mathbb{R}^d)$ factorize as $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$?

- In 1978, Rubel, Squires, and Taylor, showed that $\mathcal{D}(\mathbb{R}^d)$ has the weak factorization property, namely,

$$\mathcal{D}(\mathbb{R}^d) = \operatorname{span}(\mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d))$$

- If $d \geq 3$, they also showed that $\mathcal{D}(\mathbb{R}^d)$ does not have the strong factorization property.
- Dixmier and Malliavin (1979): negative answer for $d = 2$.
- Yulmukhametov (1999): in contrast $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$ holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

$$\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d) * \mathcal{S}(\mathbb{R}^d).$$

Factorization in algebras of smooth functions

- Ehrenpreis' problem (1960):

Does $\mathcal{D}(\mathbb{R}^d)$ factorize as $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$?

- In 1978, Rubel, Squires, and Taylor, showed that $\mathcal{D}(\mathbb{R}^d)$ has the weak factorization property, namely,

$$\mathcal{D}(\mathbb{R}^d) = \operatorname{span}(\mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d))$$

- If $d \geq 3$, they also showed that $\mathcal{D}(\mathbb{R}^d)$ does not have the strong factorization property.
- Dixmier and Malliavin (1979): negative answer for $d = 2$.
- Yulmukhametov (1999): in contrast $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$ holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

$$\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d) * \mathcal{S}(\mathbb{R}^d).$$

Factorization in algebras of smooth functions

- Ehrenpreis' problem (1960):

> Does $\mathcal{D}(\mathbb{R}^d)$ factorize as $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$?

- In 1978, Rubel, Squires, and Taylor, showed that $\mathcal{D}(\mathbb{R}^d)$ has the weak factorization property, namely,

$$\mathcal{D}(\mathbb{R}^d) = \operatorname{span}(\mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d))$$

- If $d \geq 3$, they also showed that $\mathcal{D}(\mathbb{R}^d)$ does not have the strong factorization property.
- Dixmier and Malliavin (1979): negative answer for $d = 2$.
- Yulmukhametov (1999): in contrast $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$ holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

$$\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d) * \mathcal{S}(\mathbb{R}^d).$$

Factorization in algebras of smooth functions

- Ehrenpreis' problem (1960):

Does $\mathcal{D}(\mathbb{R}^d)$ factorize as $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$?

- In 1978, Rubel, Squires, and Taylor, showed that $\mathcal{D}(\mathbb{R}^d)$ has the weak factorization property, namely,

$$\mathcal{D}(\mathbb{R}^d) = \operatorname{span}(\mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d))$$

- If $d \geq 3$, they also showed that $\mathcal{D}(\mathbb{R}^d)$ does not have the strong factorization property.
- Dixmier and Malliavin (1979): negative answer for $d = 2$.
- Yulmukhametov (1999): in contrast $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$ holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

$$\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d) * \mathcal{S}(\mathbb{R}^d).$$

Factorization in algebras of smooth functions

- Ehrenpreis' problem (1960):

Does $\mathcal{D}(\mathbb{R}^d)$ factorize as $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$?

- In 1978, Rubel, Squires, and Taylor, showed that $\mathcal{D}(\mathbb{R}^d)$ has the weak factorization property, namely,

$$\mathcal{D}(\mathbb{R}^d) = \operatorname{span}(\mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d))$$

- If $d \geq 3$, they also showed that $\mathcal{D}(\mathbb{R}^d)$ does not have the strong factorization property.
- Dixmier and Malliavin (1979): negative answer for $d = 2$.
- Yulmukhametov (1999): in contrast $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$ holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

$$\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d) * \mathcal{S}(\mathbb{R}^d).$$

Factorization in algebras of smooth functions

- Ehrenpreis' problem (1960):

Does $\mathcal{D}(\mathbb{R}^d)$ factorize as $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$?

- In 1978, Rubel, Squires, and Taylor, showed that $\mathcal{D}(\mathbb{R}^d)$ has the weak factorization property, namely,

$$\mathcal{D}(\mathbb{R}^d) = \operatorname{span}(\mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d))$$

- If $d \geq 3$, they also showed that $\mathcal{D}(\mathbb{R}^d)$ does not have the strong factorization property.
- Dixmier and Malliavin (1979): negative answer for $d = 2$.
- Yulmukhametov (1999): in contrast $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$ holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

$$\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d) * \mathcal{S}(\mathbb{R}^d).$$

Factorization on Lie groups

- Let G be a real connected Lie group.
- Dixmier and Malliavin showed (1979) that

$$\mathcal{D}(G) = \operatorname{span}(\mathcal{D}(G) * \mathcal{D}(G))$$

  and, when additionally G is nilpotent,

$$\mathcal{S}(G) = \mathcal{S}(G) * \mathcal{S}(G).$$

  (hereafter: convolution = left convolution)
- Let E be a locally convex Hausdorff space (lcHs) and denote as $\mathrm{GL}(E)$ its group of isomorphisms.
- A group homomorphism $\pi : G \to \mathrm{GL}(E)$ such that

$$G \times E \to E, \quad (g, e) \mapsto \pi(g)e$$

  is separately continuous is a representation of G on E.
- We call $e \in E$ a smooth vector if its orbit mapping

$$G \to E \quad g \mapsto \pi(g)e, \quad \text{belongs to } C^\infty(G; E).$$

- E^∞ is the subspace of smooth vectors.

Factorization on Lie groups

- Let G be a real connected Lie group.
- Dixmier and Malliavin showed (1979) that

$$\mathcal{D}(G) = \operatorname{span}(\mathcal{D}(G) * \mathcal{D}(G))$$

  and, when additionally G is nilpotent,

$$\mathcal{S}(G) = \mathcal{S}(G) * \mathcal{S}(G).$$

  (hereafter: convolution = left convolution)
- Let E be a locally convex Hausdorff space (lcHs) and denote as $\mathrm{GL}(E)$ its group of isomorphisms.
- A group homomorphism $\pi : G \to \mathrm{GL}(E)$ such that

$$G \times E \to E, \quad (g, e) \mapsto \pi(g)e$$

  is separately continuous is a representation of G on E.
- We call $e \in E$ a smooth vector if its orbit mapping

$$G \to E \quad g \mapsto \pi(g)e, \quad \text{belongs to } C^\infty(G; E).$$

- E^∞ is the subspace of smooth vectors.

Factorization on Lie groups

- Let G be a real connected Lie group.
- Dixmier and Malliavin showed (1979) that
$$\mathcal{D}(G) = \text{span}(\mathcal{D}(G) * \mathcal{D}(G))$$
and, when additionally G is nilpotent,
$$\mathcal{S}(G) = \mathcal{S}(G) * \mathcal{S}(G).$$
(hereafter: convolution = left convolution)
- Let E be a locally convex Hausdorff space (lcHs) and denote as $\mathrm{GL}(E)$ its group of isomorphisms.
- A group homomorphism $\pi : G \to \mathrm{GL}(E)$ such that
$$G \times E \to E, \quad (g, e) \mapsto \pi(g)e$$
is separately continuous is a representation of G on E.
- We call $e \in E$ a smooth vector if its orbit mapping
$$G \to E \quad g \mapsto \pi(g)e, \quad \text{belongs to } C^\infty(G; E).$$
- E^∞ is the subspace of smooth vectors.

Dixmier-Malliavin factorization theorems

- A representation of G on a sequentially complete lcHs E induces an action of the convolution algebra $\mathcal{D}(G)$ on the smooth vectors,

$$(f,e) \mapsto \Pi(f)e, \quad \mathcal{D}(G) \times E^{\infty} \to E^{\infty}, \quad \text{where}$$

$$\boxed{\Pi(f)e = \int_G f(g)\pi(g)e \, \mathrm{d}\, g \in E}$$

- If E is Banach and the representation is bounded, the action extends to $\mathcal{S}(G)$ and we can regard E^{∞} as a module over $\mathcal{S}(G)$.

Theorem

If E is a Fréchet space, E^{∞} has the weak factorization property w.r.t. $\mathcal{D}(G)$, that is, $\boxed{E^{\infty} = \operatorname{span}(\Pi(\mathcal{D}(G))E^{\infty})}$.

Theorem

If E is a Hilbert space, the representation is unitary, and G is nilpotent, then E^{∞} has the strong factorization property w.r.t. $\mathcal{S}(G)$.

Dixmier-Malliavin factorization theorems

- A representation of G on a sequentially complete lcHs E induces an action of the convolution algebra $\mathcal{D}(G)$ on the smooth vectors,

$$(f, e) \mapsto \Pi(f)e, \quad \mathcal{D}(G) \times E^{\infty} \to E^{\infty}, \quad \text{where}$$

$$\boxed{\Pi(f)e = \int_G f(g)\pi(g)e \, \mathrm{d}g \in E}$$

- If E is Banach and the representation is bounded, the action extends to $\mathcal{S}(G)$ and we can regard E^{∞} as a module over $\mathcal{S}(G)$.

Theorem

If E is a Fréchet space, E^{∞} has the weak factorization property w.r.t. $\mathcal{D}(G)$, that is, $\boxed{E^{\infty} = \operatorname{span}(\Pi(\mathcal{D}(G))E^{\infty})}$.

Theorem

If E is a Hilbert space, the representation is unitary, and G is nilpotent, then E^{∞} has the strong factorization property w.r.t. $\mathcal{S}(G)$.

Dixmier-Malliavin factorization theorems

- A representation of G on a sequentially complete lcHs E induces an action of the convolution algebra $\mathcal{D}(G)$ on the smooth vectors,

$$(f,e) \mapsto \Pi(f)e, \quad \mathcal{D}(G) \times E^{\infty} \to E^{\infty}, \quad \text{where}$$

$$\boxed{\Pi(f)e = \int_G f(g)\pi(g)e \, \mathrm{d}\, g \in E}$$

- If E is Banach and the representation is bounded, the action extends to $\mathcal{S}(G)$ and we can regard E^{∞} as a module over $\mathcal{S}(G)$.

Theorem

If E is a Fréchet space, E^{∞} has the weak factorization property w.r.t. $\mathcal{D}(G)$, that is, $\boxed{E^{\infty} = \operatorname{span}(\Pi(\mathcal{D}(G))E^{\infty})}$.

Theorem

If E is a Hilbert space, the representation is unitary, and G is nilpotent, then E^{∞} has the strong factorization property w.r.t. $\mathcal{S}(G)$.

Dixmier-Malliavin factorization theorems

- A representation of G on a sequentially complete lcHs E induces an action of the convolution algebra $\mathcal{D}(G)$ on the smooth vectors,

$$(f, e) \mapsto \Pi(f)e, \quad \mathcal{D}(G) \times E^\infty \to E^\infty, \quad \text{where}$$

$$\boxed{\Pi(f)e = \int_G f(g)\pi(g)e \, \mathrm{d}\, g \in E}$$

- If E is Banach and the representation is bounded, the action extends to $\mathcal{S}(G)$ and we can regard E^∞ as a module over $\mathcal{S}(G)$.

Theorem

If E is a Fréchet space, E^∞ has the weak factorization property w.r.t. $\mathcal{D}(G)$, that is, $\boxed{E^\infty = \operatorname{span}(\Pi(\mathcal{D}(G))E^\infty)}$.

Theorem

If E is a Hilbert space, the representation is unitary, and G is nilpotent, then E^∞ has the strong factorization property w.r.t. $\mathcal{S}(G)$.

Analytic factorization of Lie group representations

- $e \in E$ is an analytic vector if $g \mapsto \pi(g)e$ is an analytic mapping.
- E^ω: subspace of analytic vectors.
- A representation is called an F-representation if
  - E is a Fréchet space;
  - there is a basis of continuous seminorms $(p_n)_{n\in\mathbb{N}}$ such that for each n the action $G \times (E, p_n) \to (E, p_n)$ is continuous.
- For F-representations, we get an action of the algebra of exponentially rapidly decreasing analytic functions $\mathcal{A}(G)$ on E^ω.

Theorem (Gimperlein, Krötz, and Lienau (2012))

For F-representations, E^ω has the weak factorization property w.r.t. $\mathcal{A}(G)$, that is, $\boxed{E^\omega = \operatorname{span}(\Pi(\mathcal{A}(G))E^\omega)}$.

Conjecture

They have conjectured that one might even have $\boxed{E^\omega = \Pi(\mathcal{A}(G))E^\omega}$.

Analytic factorization of Lie group representations

- $e \in E$ is an analytic vector if $g \mapsto \pi(g)e$ is an analytic mapping.
- E^ω: subspace of analytic vectors.
- A representation is called an F-representation if
  - E is a Fréchet space;
  - there is a basis of continuous seminorms $(p_n)_{n\in\mathbb{N}}$ such that for each n the action $G \times (E, p_n) \to (E, p_n)$ is continuous.
- For F-representations, we get an action of the algebra of exponentially rapidly decreasing analytic functions $\mathcal{A}(G)$ on E^ω.

Theorem (Gimperlein, Krötz, and Lienau (2012))

For F-representations, E^ω has the weak factorization property w.r.t. $\mathcal{A}(G)$, that is, $\boxed{E^\omega = \operatorname{span}(\Pi(\mathcal{A}(G))E^\omega)}$.

Conjecture

They have conjectured that one might even have $\boxed{E^\omega = \Pi(\mathcal{A}(G))E^\omega}$.

Analytic factorization of Lie group representations

- $e \in E$ is an analytic vector if $g \mapsto \pi(g)e$ is an analytic mapping.
- E^ω: subspace of analytic vectors.
- A representation is called an F-representation if
  - E is a Fréchet space;
  - there is a basis of continuous seminorms $(p_n)_{n\in\mathbb{N}}$ such that for each n the action $G \times (E, p_n) \to (E, p_n)$ is continuous.
- For F-representations, we get an action of the algebra of exponentially rapidly decreasing analytic functions $\mathcal{A}(G)$ on E^ω.

Theorem (Gimperlein, Krötz, and Lienau (2012))

For F-representations, E^ω has the weak factorization property w.r.t. $\mathcal{A}(G)$, that is, $\boxed{E^\omega = \operatorname{span}(\Pi(\mathcal{A}(G))E^\omega)}$.

Conjecture

They have conjectured that one might even have $\boxed{E^\omega = \Pi(\mathcal{A}(G))E^\omega}$.

Analytic factorization of Lie group representations

- $e \in E$ is an analytic vector if $g \mapsto \pi(g)e$ is an analytic mapping.
- E^ω: subspace of analytic vectors.
- A representation is called an F-representation if
  - E is a Fréchet space;
  - there is a basis of continuous seminorms $(p_n)_{n\in\mathbb{N}}$ such that for each n the action $G \times (E, p_n) \to (E, p_n)$ is continuous.
- For F-representations, we get an action of the algebra of exponentially rapidly decreasing analytic functions $\mathcal{A}(G)$ on E^ω.

Theorem (Gimperlein, Krötz, and Lienau (2012))

For F-representations, E^ω has the weak factorization property w.r.t. $\mathcal{A}(G)$, that is, $\boxed{E^\omega = \operatorname{span}(\Pi(\mathcal{A}(G))E^\omega)}$.

Conjecture

They have conjectured that one might even have $\boxed{E^\omega = \Pi(\mathcal{A}(G))E^\omega}$.

Analytic factorization of Lie group representations

- $e \in E$ is an analytic vector if $g \mapsto \pi(g)e$ is an analytic mapping.
- E^ω: subspace of analytic vectors.
- A representation is called an F-representation if
  - E is a Fréchet space;
  - there is a basis of continuous seminorms $(p_n)_{n\in\mathbb{N}}$ such that for each n the action $G \times (E, p_n) \to (E, p_n)$ is continuous.
- For F-representations, we get an action of the algebra of exponentially rapidly decreasing analytic functions $\mathcal{A}(G)$ on E^ω.

Theorem (Gimperlein, Krötz, and Lienau (2012))

For F-representations, E^ω has the weak factorization property w.r.t. $\mathcal{A}(G)$, that is, $\boxed{E^\omega = \operatorname{span}(\Pi(\mathcal{A}(G))E^\omega)}$.

Conjecture

They have conjectured that one might even have $\boxed{E^\omega = \Pi(\mathcal{A}(G))E^\omega}$.

Analytic factorization of Lie group representations

- $e \in E$ is an analytic vector if $g \mapsto \pi(g)e$ is an analytic mapping.
- E^ω: subspace of analytic vectors.
- A representation is called an F-representation if
  - E is a Fréchet space;
  - there is a basis of continuous seminorms $(p_n)_{n\in\mathbb{N}}$ such that for each n the action $G \times (E, p_n) \to (E, p_n)$ is continuous.
- For F-representations, we get an action of the algebra of exponentially rapidly decreasing analytic functions $\mathcal{A}(G)$ on E^ω.

Theorem (Gimperlein, Krötz, and Lienau (2012))

For F-representations, E^ω has the weak factorization property w.r.t. $\mathcal{A}(G)$, that is, $\boxed{E^\omega = \operatorname{span}(\Pi(\mathcal{A}(G))E^\omega)}$.

Conjecture

They have conjectured that one might even have $\boxed{E^\omega = \Pi(\mathcal{A}(G))E^\omega}$.

Analytic factorization for $(\mathbb{R}^d, +)$

The convolution algebra $\mathcal{A}(\mathbb{R}^d)$ consists of real analytic functions f admitting holomorphic extension to $\mathbb{R}^d + i]-h, h[^d$ for some $h > 0$ and satisfying

$$\sup_{|\operatorname{Im} z| \leq h} e^{n|\operatorname{Re} z|}|f(z)| < \infty, \qquad \text{for each } n \in \mathbb{N}.$$

Theorem (Debrouwere, Prangoski, and V. (2021))

For F-representations of $\mathbb{R}^d$, E^ω has the strong factorization property w.r.t. $\mathcal{A}(\mathbb{R}^d)$, that is, $\boxed{E^\omega = \Pi(\mathcal{A}(\mathbb{R}^d))E^\omega}$.

Our results hold for more general representations than F-representations:

- projective generalized proto-Banach representations;
- inductive generalized proto-Banach representations.

Analytic factorization for $(\mathbb{R}^d, +)$

The convolution algebra $\mathcal{A}(\mathbb{R}^d)$ consists of real analytic functions f admitting holomorphic extension to $\mathbb{R}^d + i]-h, h[^d$ for some $h > 0$ and satisfying

$$\sup_{|\operatorname{Im} z| \leq h} e^{n|\operatorname{Re} z|}|f(z)| < \infty, \qquad \text{for each } n \in \mathbb{N}.$$

Theorem (Debrouwere, Prangoski, and V. (2021))

For F-representations of $\mathbb{R}^d$, E^ω has the strong factorization property w.r.t. $\mathcal{A}(\mathbb{R}^d)$, that is, $\boxed{E^\omega = \Pi(\mathcal{A}(\mathbb{R}^d))E^\omega}$.

Our results hold for more general representations than F-representations:

- projective generalized proto-Banach representations;
- inductive generalized proto-Banach representations.

Analytic factorization for $(\mathbb{R}^d, +)$

The convolution algebra $\mathcal{A}(\mathbb{R}^d)$ consists of real analytic functions f admitting holomorphic extension to $\mathbb{R}^d + i]-h, h[^d$ for some $h > 0$ and satisfying

$$\sup_{|\operatorname{Im} z| \leq h} e^{n|\operatorname{Re} z|}|f(z)| < \infty, \qquad \text{for each } n \in \mathbb{N}.$$

Theorem (Debrouwere, Prangoski, and V. (2021))

For F-representations of $\mathbb{R}^d$, E^ω has the strong factorization property w.r.t. $\mathcal{A}(\mathbb{R}^d)$, that is, $\boxed{E^\omega = \Pi(\mathcal{A}(\mathbb{R}^d))E^\omega}$.

Our results hold for more general representations than F-representations:

- projective generalized proto-Banach representations;
- inductive generalized proto-Banach representations.

Projective and inductive generalized proto-Banach representations

Definition

A representation (π, E) is said to be a projective generalized proto-Banach representation if

$$\forall p \in \operatorname{csn}(E)\, \exists q_p \in \operatorname{csn}(E)\, \exists \kappa_p > 0\, \forall x \in \mathbb{R}^d\, \forall e \in E \ :$$
$$p(\pi(x)e) \leq e^{\kappa_p |x|} q_p(e)$$

$\mathfrak{B}(E)$ stands for the collection of non-empty absolutely convex closed bounded subsets of E and for $B \in \mathfrak{B}(E)$ we denote by E_B the subspace of E spanned by B.

Definition

(π, E) is an inductive generalized proto-Banach representation if

$$\forall B \in \mathfrak{B}(E)\, \exists A_B \in \mathfrak{B}(E)\, \exists \kappa_B > 0\, \forall x \in \mathbb{R}^d\, \forall e \in E_B \ :$$
$$\|\pi(x)e\|_{E_{A_B}} \leq e^{\kappa_B |x|} \|e\|_{E_B}.$$

Projective and inductive generalized proto-Banach representations

Definition

A representation (π, E) is said to be a projective generalized proto-Banach representation if

$$\forall p \in \operatorname{csn}(E)\, \exists q_p \in \operatorname{csn}(E)\, \exists \kappa_p > 0\, \forall x \in \mathbb{R}^d\, \forall e \in E \ :$$
$$p(\pi(x)e) \leq e^{\kappa_p |x|} q_p(e)$$

$\mathfrak{B}(E)$ stands for the collection of non-empty absolutely convex closed bounded subsets of E and for $B \in \mathfrak{B}(E)$ we denote by E_B the subspace of E spanned by B.

Definition

(π, E) is an inductive generalized proto-Banach representation if

$$\forall B \in \mathfrak{B}(E)\, \exists A_B \in \mathfrak{B}(E)\, \exists \kappa_B > 0\, \forall x \in \mathbb{R}^d\, \forall e \in E_B \ :$$
$$\|\pi(x)e\|_{E_{A_B}} \leq e^{\kappa_B |x|} \|e\|_{E_B}.$$

Projective and inductive generalized proto-Banach representations

Definition

A representation (π, E) is said to be a projective generalized proto-Banach representation if

$$\forall p \in \operatorname{csn}(E)\, \exists q_p \in \operatorname{csn}(E)\, \exists \kappa_p > 0\, \forall x \in \mathbb{R}^d\, \forall e \in E\ :$$
$$p(\pi(x)e) \leq e^{\kappa_p |x|} q_p(e)$$

$\mathfrak{B}(E)$ stands for the collection of non-empty absolutely convex closed bounded subsets of E and for $B \in \mathfrak{B}(E)$ we denote by E_B the subspace of E spanned by B.

Definition

(π, E) is an inductive generalized proto-Banach representation if

$$\forall B \in \mathfrak{B}(E)\, \exists A_B \in \mathfrak{B}(E)\, \exists \kappa_B > 0\, \forall x \in \mathbb{R}^d\, \forall e \in E_B\ :$$
$$\|\pi(x)e\|_{E_{A_B}} \leq e^{\kappa_B |x|} \|e\|_{E_B}.$$

Projective and inductive generalized proto-Banach representations

Definition

A representation (π, E) is said to be a projective generalized proto-Banach representation if

$$\forall p \in \operatorname{csn}(E)\, \exists q_p \in \operatorname{csn}(E)\, \exists \kappa_p > 0\, \forall x \in \mathbb{R}^d\, \forall e \in E \;:$$
$$p(\pi(x)e) \leq e^{\kappa_p |x|} q_p(e)$$

$\mathfrak{B}(E)$ stands for the collection of non-empty absolutely convex closed bounded subsets of E and for $B \in \mathfrak{B}(E)$ we denote by E_B the subspace of E spanned by B.

Definition

(π, E) is an inductive generalized proto-Banach representation if

$$\forall B \in \mathfrak{B}(E)\, \exists A_B \in \mathfrak{B}(E)\, \exists \kappa_B > 0\, \forall x \in \mathbb{R}^d\, \forall e \in E_B \;:$$
$$\|\pi(x)e\|_{E_{A_B}} \leq e^{\kappa_B |x|} \|e\|_{E_B}.$$

Denjoy-Carleman classes

Consider a log-convex sequence $M = (M_p)_p$ of positive numbers and set $m_p = M_p/M_{p-1}$. We assume:

$(M.2)$ there are $C_0, H > 0$ such that $M_{p+q} \leq C_0 H^{p+q} M_p M_q$;

$(M.2)^*$ there are $Q, p_0 \in \mathbb{N}$ such that $2m_p \leq m_{Qp}$ for all $p \geq p_0$.

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$.

- a vector $e \in E$ is ultradifferentiable of class $[M]$ if its orbit mapping w.r.t. the representation is (bornologically) ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
- $E^{[M]}$ denotes the space of ultradifferentiable vectors of class $[M]$ of a representation.
- Note that $E^\omega = E^{\{p!\}}$.

Denjoy-Carleman classes

Consider a log-convex sequence $M = (M_p)_p$ of positive numbers and set $m_p = M_p/M_{p-1}$. We assume:

$(M.2)$ there are $C_0, H > 0$ such that $M_{p+q} \leq C_0 H^{p+q} M_p M_q$;

$(M.2)^*$ there are $Q, p_0 \in \mathbb{N}$ such that $2m_p \leq m_{Qp}$ for all $p \geq p_0$.

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$.

- a vector $e \in E$ is ultradifferentiable of class $[M]$ if its orbit mapping w.r.t. the representation is (bornologically) ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
- $E^{[M]}$ denotes the space of ultradifferentiable vectors of class $[M]$ of a representation.
- Note that $E^\omega = E^{\{p!\}}$.

Denjoy-Carleman classes

Consider a log-convex sequence $M = (M_p)_p$ of positive numbers and set $m_p = M_p/M_{p-1}$. We assume:

$(M.2)$ there are $C_0, H > 0$ such that $M_{p+q} \leq C_0 H^{p+q} M_p M_q$;

$(M.2)^*$ there are $Q, p_0 \in \mathbb{N}$ such that $2m_p \leq m_{Qp}$ for all $p \geq p_0$.

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$.

- a vector $e \in E$ is ultradifferentiable of class $[M]$ if its orbit mapping w.r.t. the representation is (bornologically) ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
- $E^{[M]}$ denotes the space of ultradifferentiable vectors of class $[M]$ of a representation.
- Note that $E^\omega = E^{\{p!\}}$.

Denjoy-Carleman classes

Consider a log-convex sequence $M = (M_p)_p$ of positive numbers and set $m_p = M_p/M_{p-1}$. We assume:

$(M.2)$ there are $C_0, H > 0$ such that $M_{p+q} \leq C_0 H^{p+q} M_p M_q$;

$(M.2)^*$ there are $Q, p_0 \in \mathbb{N}$ such that $2m_p \leq m_{Qp}$ for all $p \geq p_0$.

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$.

- a vector $e \in E$ is ultradifferentiable of class $[M]$ if its orbit mapping w.r.t. the representation is (bornologically) ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
- $E^{[M]}$ denotes the space of ultradifferentiable vectors of class $[M]$ of a representation.
- Note that $E^\omega = E^{\{p!\}}$.

Denjoy-Carleman classes

Consider a log-convex sequence $M = (M_p)_p$ of positive numbers and set $m_p = M_p/M_{p-1}$. We assume:

$(M.2)$ there are $C_0, H > 0$ such that $M_{p+q} \leq C_0 H^{p+q} M_p M_q$;

$(M.2)^*$ there are $Q, p_0 \in \mathbb{N}$ such that $2m_p \leq m_{Qp}$ for all $p \geq p_0$.

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$.

- a vector $e \in E$ is ultradifferentiable of class $[M]$ if its orbit mapping w.r.t. the representation is (bornologically) ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
- $E^{[M]}$ denotes the space of ultradifferentiable vectors of class $[M]$ of a representation.
- Note that $E^\omega = E^{\{p!\}}$.

Denjoy-Carleman classes

Consider a log-convex sequence $M = (M_p)_p$ of positive numbers and set $m_p = M_p/M_{p-1}$. We assume:

$(M.2)$ there are $C_0, H > 0$ such that $M_{p+q} \leq C_0 H^{p+q} M_p M_q$;

$(M.2)^*$ there are $Q, p_0 \in \mathbb{N}$ such that $2m_p \leq m_{Qp}$ for all $p \geq p_0$.

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$.

- a vector $e \in E$ is ultradifferentiable of class $[M]$ if its orbit mapping w.r.t. the representation is (bornologically) ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
- $E^{[M]}$ denotes the space of ultradifferentiable vectors of class $[M]$ of a representation.
- Note that $E^\omega = E^{\{p!\}}$.

Factorization theorem for ultradifferentiable vectors

For $h > 0$, we define the Fréchet space

$$\mathcal{K}^{M,h}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{h^{|\alpha|} |\partial^\alpha \varphi(x)| e^{n|x|}}{M_{|\alpha|}} < \infty, \quad \forall n \in \mathbb{N}\}.$$

We set

$$\mathcal{K}^{(M)}(\mathbb{R}^d) = \varprojlim_{h \to \infty} \mathcal{K}^{M,h}(\mathbb{R}^d) \qquad \text{and} \qquad \mathcal{K}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{h \to 0^+} \mathcal{K}^{M,h}(\mathbb{R}^d).$$

If $M_p = p!$, then $\mathcal{A}(\mathbb{R}^d) = \mathcal{K}^{\{M\}}(\mathbb{R}^d)$.

Theorem (Debrouwere, Prangoski, and V. (2021))

Let (π, E) be either a projective or an inductive generalized proto-Banach representation of $(\mathbb{R}^d, +)$ on a sequentially complete lcHs E. Then, $E^{[M]}$ has the strong factorization property w.r.t. $\mathcal{K}^{[M]}(\mathbb{R}^d)$

$$\boxed{E^{[M]} = \Pi(\mathcal{K}^{[M]}(\mathbb{R}^d))E^{[M]}}.$$

Factorization theorem for ultradifferentiable vectors

For $h > 0$, we define the Fréchet space

$$\mathcal{K}^{M,h}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{h^{|\alpha|}|\partial^\alpha \varphi(x)| e^{n|x|}}{M_{|\alpha|}} < \infty, \quad \forall n \in \mathbb{N}\}.$$

We set

$$\mathcal{K}^{(M)}(\mathbb{R}^d) = \varprojlim_{h \to \infty} \mathcal{K}^{M,h}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{K}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{h \to 0^+} \mathcal{K}^{M,h}(\mathbb{R}^d).$$

If $M_p = p!$, then $\mathcal{A}(\mathbb{R}^d) = \mathcal{K}^{\{M\}}(\mathbb{R}^d)$.

Theorem (Debrouwere, Prangoski, and V. (2021))

Let (π, E) be either a projective or an inductive generalized proto-Banach representation of $(\mathbb{R}^d, +)$ on a sequentially complete lcHs E. Then, $E^{[M]}$ has the strong factorization property w.r.t. $\mathcal{K}^{[M]}(\mathbb{R}^d)$

$$\boxed{E^{[M]} = \Pi(\mathcal{K}^{[M]}(\mathbb{R}^d))E^{[M]}}.$$

Factorization theorem for ultradifferentiable vectors

For $h > 0$, we define the Fréchet space

$$\mathcal{K}^{M,h}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{h^{|\alpha|}|\partial^\alpha \varphi(x)| e^{n|x|}}{M_{|\alpha|}} < \infty, \quad \forall n \in \mathbb{N}\}.$$

We set

$$\mathcal{K}^{(M)}(\mathbb{R}^d) = \varprojlim_{h \to \infty} \mathcal{K}^{M,h}(\mathbb{R}^d) \qquad \text{and} \qquad \mathcal{K}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{h \to 0^+} \mathcal{K}^{M,h}(\mathbb{R}^d).$$

If $M_p = p!$, then $\mathcal{A}(\mathbb{R}^d) = \mathcal{K}^{\{M\}}(\mathbb{R}^d)$.

Theorem (Debrouwere, Prangoski, and V. (2021))

Let (π, E) be either a projective or an inductive generalized proto-Banach representation of $(\mathbb{R}^d, +)$ on a sequentially complete lcHs E. Then, $E^{[M]}$ has the strong factorization property w.r.t. $\mathcal{K}^{[M]}(\mathbb{R}^d)$

$$\boxed{E^{[M]} = \Pi(\mathcal{K}^{[M]}(\mathbb{R}^d))E^{[M]}}.$$

Factorization theorem for ultradifferentiable vectors

For $h > 0$, we define the Fréchet space

$$\mathcal{K}^{M,h}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \,|\, \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in\mathbb{R}^d} \frac{h^{|\alpha|}|\partial^\alpha\varphi(x)|e^{n|x|}}{M_{|\alpha|}} < \infty, \quad \forall n \in \mathbb{N}\}.$$

We set

$$\mathcal{K}^{(M)}(\mathbb{R}^d) = \varprojlim_{h\to\infty} \mathcal{K}^{M,h}(\mathbb{R}^d) \qquad \text{and} \qquad \mathcal{K}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{h\to 0^+} \mathcal{K}^{M,h}(\mathbb{R}^d).$$

If $M_p = p!$, then $\mathcal{A}(\mathbb{R}^d) = \mathcal{K}^{\{M\}}(\mathbb{R}^d)$.

Theorem (Debrouwere, Prangoski, and V. (2021))

Let (π, E) be either a projective or an inductive generalized proto-Banach representation of $(\mathbb{R}^d, +)$ on a sequentially complete lcHs E. Then, $E^{[M]}$ has the strong factorization property w.r.t. $\mathcal{K}^{[M]}(\mathbb{R}^d)$

$$\boxed{E^{[M]} = \Pi(\mathcal{K}^{[M]}(\mathbb{R}^d))E^{[M]}}.$$

Factorization of modules of ultradifferentiable functions

- Our factorization theorem implies the strong factorization property for many concrete families of modules of ultradifferentiable functions.

Example:

- Let $\omega : \mathbb{R}^d \to (0, \infty)$ be a continuous weight function satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L^\infty_{loc}(\mathbb{R}^d).$$

- Consider $E = L^p_\omega = \{f |\, \omega \cdot f \in L^p(\mathbb{R})\}$ if $1 \leq p < \infty$.
- The ultradifferentiable vectors are (w.r.t. regular representation)

$$E^{(M)} = \mathcal{D}_E^{(M)} = \varprojlim_{h \to \infty} \mathcal{D}_E^{\{M\},h} \quad \text{and} \quad E^{\{M\}} = \mathcal{D}_E^{\{M\}} = \varinjlim_{h \to 0^+} \mathcal{D}_E^{\{M\},h},$$

$$\mathcal{D}_E^{\{M\},h} = \{\varphi \in C^\infty(\mathbb{R}^d) |\, \sup_\alpha h^{|\alpha|} \|\varphi^{(\alpha)}\|_E / M_{|\alpha|} < \infty\}.$$

We have: $\mathcal{D}_E^{[M]} = \mathcal{K}^{[M]}(\mathbb{R}^d) * \mathcal{D}_E^{[M]}$

Factorization of modules of ultradifferentiable functions

- Our factorization theorem implies the strong factorization property for many concrete families of modules of ultradifferentiable functions.

Example:

- Let $\omega : \mathbb{R}^d \to (0, \infty)$ be a continuous weight function satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L^{\infty}_{loc}(\mathbb{R}^d).$$

- Consider $E = L^p_\omega = \{f | \, \omega \cdot f \in L^p(\mathbb{R})\}$ if $1 \leq p < \infty$.
- The ultradifferentiable vectors are (w.r.t. regular representation)

$$E^{(M)} = \mathcal{D}^{(M)}_E = \varprojlim_{h \to \infty} \mathcal{D}^{\{M\},h}_E \quad \text{and} \quad E^{\{M\}} = \mathcal{D}^{\{M\}}_E = \varinjlim_{h \to 0^+} \mathcal{D}^{\{M\},h}_E,$$

$$\mathcal{D}^{\{M\},h}_E = \{\varphi \in C^\infty(\mathbb{R}^d) | \sup_\alpha h^{|\alpha|} \|\varphi^{(\alpha)}\|_E / M_{|\alpha|} < \infty\}.$$

We have: $\mathcal{D}^{[M]}_E = \mathcal{K}^{[M]}(\mathbb{R}^d) * \mathcal{D}^{[M]}_E$

- Our factorization theorem implies the strong factorization property for many concrete families of modules of ultradifferentiable functions.

Example:

- Let $\omega : \mathbb{R}^d \to (0, \infty)$ be a continuous weight function satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L^\infty_{loc}(\mathbb{R}^d).$$

- Consider $E = L^p_\omega = \{f \mid \omega \cdot f \in L^p(\mathbb{R})\}$ if $1 \leq p < \infty$.
- The ultradifferentiable vectors are (w.r.t. regular representation)

$$E^{(M)} = \mathcal{D}^{(M)}_E = \varprojlim_{h \to \infty} \mathcal{D}^{\{M\},h}_E \quad \text{and} \quad E^{\{M\}} = \mathcal{D}^{\{M\}}_E = \varinjlim_{h \to 0^+} \mathcal{D}^{\{M\},h}_E,$$

$$\mathcal{D}^{\{M\},h}_E = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \sup_\alpha h^{|\alpha|} \|\varphi^{(\alpha)}\|_E / M_{|\alpha|} < \infty\}.$$

We have: $\mathcal{D}^{[M]}_E = \mathcal{K}^{[M]}(\mathbb{R}^d) * \mathcal{D}^{[M]}_E$

- Our factorization theorem implies the strong factorization property for many concrete families of modules of ultradifferentiable functions.

Example:

- Let $\omega : \mathbb{R}^d \to (0, \infty)$ be a continuous weight function satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L^\infty_{loc}(\mathbb{R}^d).$$

- Consider $E = L^p_\omega = \{f |\, \omega \cdot f \in L^p(\mathbb{R})\}$ if $1 \leq p < \infty$.
- The ultradifferentiable vectors are (w.r.t. regular representation)

$$E^{(M)} = \mathcal{D}_E^{(M)} = \varprojlim_{h \to \infty} \mathcal{D}_E^{\{M\},h} \quad \text{and} \quad E^{\{M\}} = \mathcal{D}_E^{\{M\}} = \varinjlim_{h \to 0^+} \mathcal{D}_E^{\{M\},h},$$

$$\mathcal{D}_E^{\{M\},h} = \{\varphi \in C^\infty(\mathbb{R}^d) |\, \sup_\alpha h^{|\alpha|} \|\varphi^{(\alpha)}\|_E / M_{|\alpha|} < \infty\}.$$

We have: $\mathcal{D}_E^{[M]} = \mathcal{K}^{[M]}(\mathbb{R}^d) * \mathcal{D}_E^{[M]}$

Factorization of modules of ultradifferentiable functions

- Our factorization theorem implies the strong factorization property for many concrete families of modules of ultradifferentiable functions.

Example:

- Let $\omega : \mathbb{R}^d \to (0, \infty)$ be a continuous weight function satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L^\infty_{loc}(\mathbb{R}^d).$$

- Consider $E = L^p_\omega = \{f \mid \omega \cdot f \in L^p(\mathbb{R})\}$ if $1 \leq p < \infty$.
- The ultradifferentiable vectors are (w.r.t. regular representation)

$$E^{(M)} = \mathcal{D}_E^{(M)} = \varprojlim_{h \to \infty} \mathcal{D}_E^{\{M\},h} \quad \text{and} \quad E^{\{M\}} = \mathcal{D}_E^{\{M\}} = \varinjlim_{h \to 0^+} \mathcal{D}_E^{\{M\},h},$$

$$\mathcal{D}_E^{\{M\},h} = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \sup_\alpha h^{|\alpha|} \|\varphi^{(\alpha)}\|_E / M_{|\alpha|} < \infty\}.$$

We have: $\mathcal{D}_E^{[M]} = \mathcal{K}^{[M]}(\mathbb{R}^d) * \mathcal{D}_E^{[M]}$

References

For more details, see our preprint

- A. Debrouwere, B. Prangoski, J. Vindas, *Factorization in Denjoy-Carleman classes associated to representations of* $(\mathbb{R}^d, +)$, J. Funct. Anal. 280 (2021), Article 108831.

Related works on factorization theorems for representations:

- J. Dixmier, P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. **102** (1978), 307–330.
- H. Gimperlein, B. Krötz, C. Lienau, *Analytic factorization of Lie group representations*, J. Funct. Anal. **262** (2012), 667–681.
- H. Glöckner, *Continuity of LF-algebra representations associated to representations of Lie groups,* Kyoto J. Math. 53 (2013), 567–595.