Continuously differentiable functions on compact sets

L. Frerick, L. Loosveldt, J. Wengenroth

Valladolid, June 21, 2022

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

$$\|f\|_{C^1(K)} = \|f\|_{C(K)} + \inf\{\|df\|_{C(K)} : df \text{ is a continuous derivative}\}.$$

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

$\|f\|_{C^1(K)} = \|f\|_{C(K)} + \inf\{\|df\|_{C(K)} : df \text{ is a continuous derivative}\}.$

- Examples: $C^1(\mathbb{R}^d|K)$ (Whitney),

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

$\|f\|_{C^1(K)} = \|f\|_{C(K)} + \inf\{\|df\|_{C(K)} : df \text{ is a continuous derivative}\}.$

- Examples: $C^1(\mathbb{R}^d|K)$ (Whitney), C^1-functions on embedded manifolds

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

$$\|f\|_{C^1(K)} = \|f\|_{C(K)} + \inf\{\|df\|_{C(K)} : df \text{ is a continuous derivative}\}.$$

- Examples: $C^1(\mathbb{R}^d|K)$ (Whitney), C^1-functions on embedded manifolds
- Chain rule holds

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

$$\|f\|_{C^1(K)} = \|f\|_{C(K)} + \inf\{\|df\|_{C(K)} : df \text{ is a continuous derivative}\}.$$

- Examples: $C^1(\mathbb{R}^d|K)$ (Whitney), C^1-functions on embedded manifolds
- Chain rule holds
- Mean value inequality. γ rectifiable curve in K from x to y, $f \in C^1(K)$

  $$|f(x) - f(y)| \leq \|df\|_{C(K)} \mathrm{L}(\gamma)$$

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

$\|f\|_{C^1(K)} = \|f\|_{C(K)} + \inf\{\|df\|_{C(K)} : df \text{ is a continuous derivative}\}.$

- Examples: $C^1(\mathbb{R}^d|K)$ (Whitney), C^1-functions on embedded manifolds
- Chain rule holds
- Mean value inequality. γ rectifiable curve in K from x to y, $f \in C^1(K)$

$$|f(x) - f(y)| \leq \|df\|_{C(K)} \mathrm{L}(\gamma) \text{ and } f(x) - f(y) = \int_\gamma df \approx \sum_{k=1}^n \langle df(\gamma(\tau_k)), \gamma(t_k) - \gamma(t_{k-1}) \rangle$$

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

$$\|f\|_{C^1(K)} = \|f\|_{C(K)} + \inf\{\|df\|_{C(K)} : df \text{ is a continuous derivative}\}.$$

- Examples: $C^1(\mathbb{R}^d|K)$ (Whitney), C^1-functions on embedded manifolds
- Chain rule holds
- Mean value inequality. γ rectifiable curve in K from x to y, $f \in C^1(K)$

$$|f(x) - f(y)| \leq \|df\|_{C(K)} \mathrm{L}(\gamma) \text{ and } f(x) - f(y) = \int_\gamma df \approx \sum_{k=1}^n \langle df(\gamma(\tau_k)), \gamma(t_k) - \gamma(t_{k-1}) \rangle$$

Theorem.

$C^1(K)$ is complete if and only if K has finitely many connected components C which are pointwise Whitney regular, i.e., there are neighbourhoods U_x of $x \in K$ and $c_x > 0$ such that every $y \in C \cap U_x$ can be joined to x by a curve of length $\leq c_x |x-y|$.

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

$\|f\|_{C^1(K)} = \|f\|_{C(K)} + \inf\{\|df\|_{C(K)} : df \text{ is a continuous derivative}\}.$

- Examples: $C^1(\mathbb{R}^d|K)$ (Whitney), C^1-functions on embedded manifolds
- Chain rule holds
- Mean value inequality. γ rectifiable curve in K from x to y, $f \in C^1(K)$

$$|f(x) - f(y)| \leq \|df\|_{C(K)} L(\gamma) \text{ and } f(x) - f(y) = \int_\gamma df \approx \sum_{k=1}^{n} \langle df(\gamma(\tau_k)), \gamma(t_k) - \gamma(t_{k-1}) \rangle$$

Theorem.

$C^1(K)$ is complete if and only if K has finitely many connected components C which are pointwise Whitney regular, i.e., there are neighbourhoods U_x of $x \in K$ and $c_x > 0$ such that every $y \in C \cap U_x$ can be joined to x by a curve of length $\leq c_x |x-y|$.

Sufficiency from mean value inequality. Necessity: Banach-Steinhaus implies

$$\varphi_y(f) = \frac{f(x)-f(y)}{|x-y|}$$

Continuously differentiable functions on $K \subseteq \mathbb{R}^d$ compact

Affine-linear approximation

$f : K \to \mathbb{R}^m$ is $C^1(K)$ if there is $df : K \to L(\mathbb{R}^d, \mathbb{R}^m)$ continuous with

$$\lim_{K \ni y \to x} \frac{|f(x) - f(y) - df(x)(x-y)|}{|x-y|} = 0 \text{ for all } x \in K.$$

$$\|f\|_{C^1(K)} = \|f\|_{C(K)} + \inf\{\|df\|_{C(K)} : df \text{ is a continuous derivative}\}.$$

- Examples: $C^1(\mathbb{R}^d|K)$ (Whitney), C^1-functions on embedded manifolds
- Chain rule holds
- Mean value inequality. γ rectifiable curve in K from x to y, $f \in C^1(K)$

$$|f(x) - f(y)| \leq \|df\|_{C(K)} \mathrm{L}(\gamma) \text{ and } f(x) - f(y) = \int_\gamma df \approx \sum_{k=1}^n \langle df(\gamma(\tau_k)), \gamma(t_k) - \gamma(t_{k-1})\rangle$$

Theorem.

$C^1(K)$ is complete if and only if K has finitely many connected components C which are pointwise Whitney regular, i.e., there are neighbourhoods U_x of $x \in K$ and $c_x > 0$ such that every $y \in C \cap U_x$ can be joined to x by a curve of length $\leq c_x|x-y|$.

Sufficiency from mean value inequality. Necessity: Banach-Steinhaus implies $|\varphi_y(f)| = \left|\frac{f(x)-f(y)}{|x-y|}\right| \leq c_x \|f\|_{C^1(K)}$ for all $f \in C^1(K)$ and $y \in K \setminus \{x\}$. Construction of suitable curves by Schwartz using Arzelá-Ascoli.

Extension of derivatives to the boundary

- If K is the closure of its interior $\mathring{K}$

  $$C^1_{int}(K) = \{f \in C(K) : f|_{\mathring{K}} \in C^1(\mathring{K}), df : \mathring{K} \to L(\mathbb{R}^d, \mathbb{R}^m) \text{ extends continuously to } K\}$$

Extension of derivatives to the boundary

- If K is the closure of its interior $\mathring{K}$

  $$C^1_{int}(K) = \{f \in C(K) : f|_{\mathring{K}} \in C^1(\mathring{K}), df : \mathring{K} \to L(\mathbb{R}^d, \mathbb{R}^m) \text{ extends continuously to } K\}$$

- With $\|f\|_{C^1_{int}(K)} = \|f\|_{C(K)} + \|df\|_{C(K)}$ this is always Banach and $C^1(K) \subseteq C^1_{int}(K)$.

Extension of derivatives to the boundary

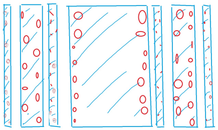
- If K is the closure of its interior $\mathring{K}$

  $$C^1_{int}(K) = \{f \in C(K) : f|_{\mathring{K}} \in C^1(\mathring{K}), df : \mathring{K} \to L(\mathbb{R}^d, \mathbb{R}^m) \text{ extends continuously to } K\}$$

- With $\|f\|_{C^1_{int}(K)} = \|f\|_{C(K)} + \|df\|_{C(K)}$ this is always Banach and $C^1(K) \subseteq C^1_{int}(K)$.
- Example (M. Sauter). $C^1(K) \neq C^1_{int}(K)$ for $K = \overline{\Omega}$ with

  $$\Omega = \Big(([0,1] \setminus \text{Cantor}) \times (0,1) \Big) \setminus \bigcup_{n \in \mathbb{N}} B_n$$

  where B_n are disjoint balls whose centres accumulate at Cantor $\times [0,1]$ and the sum of diam$(B_n) < 1/4$.



Extension of derivatives to the boundary

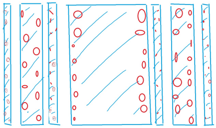
- If K is the closure of its interior $\mathring{K}$

  $$C^1_{int}(K) = \{f \in C(K) : f|_{\mathring{K}} \in C^1(\mathring{K}), df : \mathring{K} \to L(\mathbb{R}^d, \mathbb{R}^m) \text{ extends continuously to } K\}$$

- With $\|f\|_{C^1_{int}(K)} = \|f\|_{C(K)} + \|df\|_{C(K)}$ this is always Banach and $C^1(K) \subseteq C^1_{int}(K)$.
- Example (M. Sauter). $C^1(K) \neq C^1_{int}(K)$ for $K = \overline{\Omega}$ with

  $$\Omega = \Big(([0,1] \setminus \text{Cantor}) \times (0,1)\Big) \setminus \bigcup_{n\in\mathbb{N}} B_n$$

  where B_n are disjoint balls whose centres accumulate at Cantor $\times[0,1]$ and the sum of diam$(B_n) < 1/4$.



$f(x,y) = g(x)$ with Cantor's staircase function $g : [0,1] \to \mathbb{R}$.

Extension of derivatives to the boundary

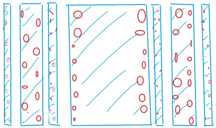
- If K is the closure of its interior $\mathring{K}$

  $$C^1_{int}(K) = \{f \in C(K) : f|_{\mathring{K}} \in C^1(\mathring{K}), df : \mathring{K} \to L(\mathbb{R}^d, \mathbb{R}^m) \text{ extends continuously to } K\}$$

- With $\|f\|_{C^1_{int}(K)} = \|f\|_{C(K)} + \|df\|_{C(K)}$ this is always Banach and $C^1(K) \subseteq C^1_{int}(K)$.
- Example (M. Sauter). $C^1(K) \neq C^1_{int}(K)$ for $K = \overline{\Omega}$ with

  $$\Omega = \Big(([0,1] \setminus \text{Cantor}) \times (0,1)\Big) \setminus \bigcup_{n \in \mathbb{N}} B_n$$

  where B_n are disjoint balls whose centres accumulate at Cantor $\times[0,1]$ and the sum of diam$(B_n) < 1/4$.



$f(x,y) = g(x)$ with Cantor's staircase function $g : [0,1] \to \mathbb{R}$. $f \in C^1_{int}(K)$, $df = 0$, and $\int_\gamma df \neq f(y) - f(x)$ for a horizontal line γ in K from x to y.

Extension of derivatives to the boundary

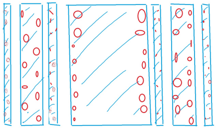
- If K is the closure of its interior $\mathring{K}$

  $$C^1_{int}(K) = \{f \in C(K) : f|_{\mathring{K}} \in C^1(\mathring{K}), df : \mathring{K} \to L(\mathbb{R}^d, \mathbb{R}^m) \text{ extends continuously to } K\}$$

- With $\|f\|_{C^1_{int}(K)} = \|f\|_{C(K)} + \|df\|_{C(K)}$ this is always Banach and $C^1(K) \subseteq C^1_{int}(K)$.
- Example (M. Sauter). $C^1(K) \neq C^1_{int}(K)$ for $K = \overline{\Omega}$ with

  $$\Omega = \Big(([0,1] \setminus \text{Cantor}) \times (0,1) \Big) \setminus \bigcup_{n \in \mathbb{N}} B_n$$

  where B_n are disjoint balls whose centres accumulate at Cantor $\times [0,1]$ and the sum of $\operatorname{diam}(B_n) < 1/4$.

  $f(x,y) = g(x)$ with Cantor's staircase function $g : [0,1] \to \mathbb{R}$. $f \in C^1_{int}(K)$, $df = 0$, and $\int_\gamma df \neq f(y) - f(x)$ for a horizontal line γ in K from x to y.
- Catastrophe: Compositions of C^1_{int}-functions need not be C^1_{int}!

Approximation.

Theorem

$C^1(\mathbb{R}^d|K)$ is dense in $C^1(K)$ for every compact set $K \subseteq \mathbb{R}^d$.

Approximation.

Theorem

$C^1(\mathbb{R}^d|K)$ is dense in $C^1(K)$ for every compact set $K \subseteq \mathbb{R}^d$.

- Standard approximation tricks (like gluing local approximation with a partition of unity or mollifying by convolution) need Whitney regularity of K.

Approximation.

Theorem

$C^1(\mathbb{R}^d|K)$ is dense in $C^1(K)$ for every compact set $K \subseteq \mathbb{R}^d$.

- Standard approximation tricks (like gluing local approximation with a partition of unity or mollifying by convolution) need Whitney regularity of K.
- Strategy: $C^1(K)$ quotient of $J^1(K) = \{(f, df) \in C(K) : df \text{ derivative of } f\}$, show density of $i : \mathscr{D}(\mathbb{R}^d) \to J^1(K)$, $\varphi \mapsto (\varphi|_K, d\varphi|_K)$ by Hahn-Banach, i.e., every continuous linear functional Φ on $J^1(K)$ which vanishes on $\{i(\varphi) : \varphi \in \mathscr{D}(\mathbb{R}^d)\}$ is zero.

Approximation.

Theorem

$C^1(\mathbb{R}^d|K)$ is dense in $C^1(K)$ for every compact set $K \subseteq \mathbb{R}^d$.

- Standard approximation tricks (like gluing local approximation with a partition of unity or mollifying by convolution) need Whitney regularity of K.
- Strategy: $C^1(K)$ quotient of $J^1(K) = \{(f, df) \in C(K) : df \text{ derivative of } f\}$, show density of $i : \mathscr{D}(\mathbb{R}^d) \to J^1(K)$, $\varphi \mapsto (\varphi|_K, d\varphi|_K)$ by Hahn-Banach, i.e., every continuous linear functional Φ on $J^1(K)$ which vanishes on $\{i(\varphi) : \varphi \in \mathscr{D}(\mathbb{R}^d)\}$ is zero.
- Hahn-Banach and Riesz-Markov-Kakutani yield measures $\nu : B(K) \to \mathbb{R}$ and $\mu : B(K) \to \mathbb{R}^d$ such that

$$\Phi(f, df) = \int_K f d\nu + \int_K \langle df, \mu \rangle \text{ and } \nu = \operatorname{div}(\mu)$$

in the sense of distributions.

Approximation.

Theorem

$C^1(\mathbb{R}^d|K)$ is dense in $C^1(K)$ for every compact set $K \subseteq \mathbb{R}^d$.

- Standard approximation tricks (like gluing local approximation with a partition of unity or mollifying by convolution) need Whitney regularity of K.
- Strategy: $C^1(K)$ quotient of $J^1(K) = \{(f, df) \in C(K) : df \text{ derivative of } f\}$, show density of $i : \mathscr{D}(\mathbb{R}^d) \to J^1(K)$, $\varphi \mapsto (\varphi|_K, d\varphi|_K)$ by Hahn-Banach, i.e., every continuous linear functional Φ on $J^1(K)$ which vanishes on $\{i(\varphi) : \varphi \in \mathscr{D}(\mathbb{R}^d)\}$ is zero.
- Hahn-Banach and Riesz-Markov-Kakutani yield measures $\nu : B(K) \to \mathbb{R}$ and $\mu : B(K) \to \mathbb{R}^d$ such that

  $$\Phi(f, df) = \int_K f d\nu + \int_K \langle df, \mu\rangle \text{ and } \nu = \mathrm{div}(\mu)$$

  in the sense of distributions.
- Stanislav K. Smirnov called such a vector measure μ a solenoidal vector charge and proved in 1993 a Choquet type decomposion into very simple solenoids comming from Lipschitz curves in K.

Approximation.

Theorem

$C^1(\mathbb{R}^d|K)$ is dense in $C^1(K)$ for every compact set $K \subseteq \mathbb{R}^d$.

- Standard approximation tricks (like gluing local approximation with a partition of unity or mollifying by convolution) need Whitney regularity of K.
- Strategy: $C^1(K)$ quotient of $J^1(K) = \{(f, df) \in C(K) : df \text{ derivative of } f\}$, show density of $i : \mathscr{D}(\mathbb{R}^d) \to J^1(K)$, $\varphi \mapsto (\varphi|_K, d\varphi|_K)$ by Hahn-Banach, i.e., every continuous linear functional Φ on $J^1(K)$ which vanishes on $\{i(\varphi) : \varphi \in \mathscr{D}(\mathbb{R}^d)\}$ is zero.
- Hahn-Banach and Riesz-Markov-Kakutani yield measures $\nu : B(K) \to \mathbb{R}$ and $\mu : B(K) \to \mathbb{R}^d$ such that

  $$\Phi(f, df) = \int_K f d\nu + \int_K \langle df, \mu \rangle \text{ and } \nu = \operatorname{div}(\mu)$$

  in the sense of distributions.
- Stanislav K. Smirnov called such a vector measure μ a solenoidal vector charge and proved in 1993 a Choquet type decomposion into very simple solenoids comming from Lipschitz curves in K.
- This decomposition can be used to show that Φ vanishes on $J^1(K)$.

Equalities in $C^1(\mathbb{R}^d|K) \subseteq C^1(K) \subseteq C^1_{int}(K)$.

Theorem.

$C(\mathbb{R}^d|K) = C^1(K)$ with equivalent norms if and only if K has only finitely many components C which are all Whitney regular, i.e., all $x, y \in C$ can be joined by a curve of length $\leq c|x - y|$.

Equalities in $C^1(\mathbb{R}^d|K) \subseteq C^1(K) \subseteq C^1_{int}(K)$.

Theorem.

$C(\mathbb{R}^d|K) = C^1(K)$ with equivalent norms if and only if K has only finitely many components C which are all Whitney regular, i.e., all $x, y \in C$ can be joined by a curve of length $\leq c|x-y|$.

Open problem: Characterize algebraic equality for $d \geq 2$.

Equalities in $C^1(\mathbb{R}^d|K) \subseteq C^1(K) \subseteq C^1_{int}(K)$.

Theorem.

$C(\mathbb{R}^d|K) = C^1(K)$ with equivalent norms if and only if K has only finitely many components C which are all Whitney regular, i.e., all $x, y \in C$ can be joined by a curve of length $\leq c|x-y|$.

Open problem: Characterize algebraic equality for $d \geq 2$.

Theorem (Whitney).

If K is the closure of its interior which is Whitney regular, then $C^1(\mathbb{R}^d|K) = C^1_{int}(K)$.

Equalities in $C^1(\mathbb{R}^d|K) \subseteq C^1(K) \subseteq C^1_{int}(K)$.

Theorem.

$C(\mathbb{R}^d|K) = C^1(K)$ with equivalent norms if and only if K has only finitely many components C which are all Whitney regular, i.e., all $x, y \in C$ can be joined by a curve of length $\leq c|x-y|$.

Open problem: Characterize algebraic equality for $d \geq 2$.

Theorem (Whitney).

If K is the closure of its interior which is Whitney regular, then $C^1(\mathbb{R}^d|K) = C^1_{int}(K)$.

Example. There are path-connected compact sets $K = \overline{\mathrm{Int}(K)}$ with $C^1(\mathbb{R}^2|K) = C^1_{int}(K)$ and $\mathrm{Int}(K)$ is not Whitney regular.

Equalities in $C^1(\mathbb{R}^d|K) \subseteq C^1(K) \subseteq C^1_{int}(K)$.

Theorem.

$C(\mathbb{R}^d|K) = C^1(K)$ with equivalent norms if and only if K has only finitely many components C which are all Whitney regular, i.e., all $x, y \in C$ can be joined by a curve of length $\leq c|x-y|$.

Open problem: Characterize algebraic equality for $d \geq 2$.

Theorem (Whitney).

If K is the closure of its interior which is Whitney regular, then $C^1(\mathbb{R}^d|K) = C^1_{int}(K)$.

Example. There are path-connected compact sets $K = \overline{\mathrm{Int}(K)}$ with $C^1(\mathbb{R}^2|K) = C^1_{int}(K)$ and $\mathrm{Int}(K)$ is not Whitney regular.

Corollary of characterization of completeness

If $K = \overline{\mathrm{Int}(K)}$ and $C^1(K) = C^1_{int}(K)$ then K has only finitely many connected components which are pointwise Whitney regular.

The one-dimensional case $K \subseteq \mathbb{R}$.

If $K = \overline{\mathrm{Int}(K)}$, then $C^1(K) = C^1_{int}(K)$ if and only if K is a finite union of closed intervals.

The one-dimensional case $K \subseteq \mathbb{R}$.

If $K = \overline{\mathrm{Int}(K)}$, then $C^1(K) = C^1_{int}(K)$ if and only if K is a finite union of closed intervals.

Theorem (Whitney).

$f \in C^1(\mathbb{R}|K) \Leftrightarrow \lim\limits_{x,y\to\xi} \frac{f(x)-f(y)}{x-y} = f'(\xi)$ for all non-isolated $\xi \in K$.

The one-dimensional case $K \subseteq \mathbb{R}$.

If $K = \overline{\mathrm{Int}(K)}$, then $C^1(K) = C^1_{int}(K)$ if and only if K is a finite union of closed intervals.

Theorem (Whitney).

$f \in C^1(\mathbb{R}|K) \Leftrightarrow \lim\limits_{x,y \to \xi} \frac{f(x)-f(y)}{x-y} = f'(\xi)$ for all non-isolated $\xi \in K$.

Definition. Every bounded connected component of $\mathbb{R} \setminus K$ is called a gap of K.

The one-dimensional case $K \subseteq \mathbb{R}$.

If $K = \overline{\mathrm{Int}(K)}$, then $C^1(K) = C^1_{int}(K)$ if and only if K is a finite union of closed intervals.

Theorem (Whitney).

$f \in C^1(\mathbb{R}|K) \Leftrightarrow \lim_{x,y\to\xi} \frac{f(x)-f(y)}{x-y} = f'(\xi)$ for all non-isolated $\xi \in K$.

Definition. Every bounded connected component of $\mathbb{R} \setminus K$ is called a gap of K.

Theorem.

$C^1(\mathbb{R}|K) = C^1(K)$ if and only if

$$\lim_{\varepsilon\to 0} \sup \left\{ \frac{\sup\{|x-y| : y \in G\}}{\mathrm{length(G)}} : G \subseteq (x-\varepsilon, x+\varepsilon) \text{ gap of } K \right\} < \infty \text{ for all } x \in K.$$

The one-dimensional case $K \subseteq \mathbb{R}$.

If $K = \overline{\mathrm{Int}(K)}$, then $C^1(K) = C^1_{int}(K)$ if and only if K is a finite union of closed intervals.

Theorem (Whitney).

$f \in C^1(\mathbb{R}|K) \Leftrightarrow \lim_{x,y\to\xi} \frac{f(x)-f(y)}{x-y} = f'(\xi)$ for all non-isolated $\xi \in K$.

Definition. Every bounded connected component of $\mathbb{R} \setminus K$ is called a gap of K.

Theorem.

$C^1(\mathbb{R}|K) = C^1(K)$ if and only if

$$\lim_{\varepsilon\to 0} \sup \left\{ \frac{\sup\{|x-y| : y \in G\}}{\mathrm{length(G)}} : G \subseteq (x-\varepsilon, x+\varepsilon) \text{ gap of } K \right\} < \infty \text{ for all } x \in K.$$

Examples.

- $C^1(K) \neq C^1(\mathbb{R}|K)$ for the Cantor set.

The one-dimensional case $K \subseteq \mathbb{R}$.

If $K = \overline{\mathrm{Int}(K)}$, then $C^1(K) = C^1_{int}(K)$ if and only if K is a finite union of closed intervals.

Theorem (Whitney).

$f \in C^1(\mathbb{R}|K) \Leftrightarrow \lim\limits_{x,y \to \xi} \frac{f(x)-f(y)}{x-y} = f'(\xi)$ for all non-isolated $\xi \in K$.

Definition. Every bounded connected component of $\mathbb{R} \setminus K$ is called a gap of K.

Theorem.

$C^1(\mathbb{R}|K) = C^1(K)$ if and only if

$$\lim_{\varepsilon \to 0} \sup \left\{ \frac{\sup\{|x-y| : y \in G\}}{\mathrm{length(G)}} : G \subseteq (x-\varepsilon, x+\varepsilon) \text{ gap of } K \right\} < \infty \text{ for all } x \in K.$$

Examples.

- $C^1(K) \neq C^1(\mathbb{R}|K)$ for the Cantor set.
- For $x_n \searrow 0$ and $K = \{0\} \cup \{x_n : n \in \mathbb{N}\}$ we have $C^1(K) = C^1(\mathbb{R}|K) \Leftrightarrow \limsup\limits_{n \to \infty} \frac{x_n}{x_n - x_{n+1}} < \infty$

The one-dimensional case $K \subseteq \mathbb{R}$.

If $K = \overline{\mathrm{Int}(K)}$, then $C^1(K) = C^1_{int}(K)$ if and only if K is a finite union of closed intervals.

Theorem (Whitney).

$f \in C^1(\mathbb{R}|K) \Leftrightarrow \lim\limits_{x,y \to \xi} \frac{f(x)-f(y)}{x-y} = f'(\xi)$ for all non-isolated $\xi \in K$.

Definition. Every bounded connected component of $\mathbb{R} \setminus K$ is called a gap of K.

Theorem.

$C^1(\mathbb{R}|K) = C^1(K)$ if and only if
$\limsup\limits_{\varepsilon \to 0} \left\{ \frac{\sup\{|x-y| : y \in G\}}{\mathrm{length(G)}} : G \subseteq (x-\varepsilon, x+\varepsilon) \text{ gap of } K \right\} < \infty$ for all $x \in K$.

Examples.

- $C^1(K) \neq C^1(\mathbb{R}|K)$ for the Cantor set.
- For $x_n \searrow 0$ and $K = \{0\} \cup \{x_n : n \in \mathbb{N}\}$ we have $C^1(K) = C^1(\mathbb{R}|K) \Leftrightarrow \limsup\limits_{n \to \infty} \frac{x_n}{x_n - x_{n+1}} < \infty$
- Yes for $x_n = a^{-n}$ with $a > 1$ but No for $x_n = n^p$ with $p < 0$.

The one-dimensional case $K \subseteq \mathbb{R}$.

If $K = \overline{\mathrm{Int}(K)}$, then $C^1(K) = C^1_{int}(K)$ if and only if K is a finite union of closed intervals.

Theorem (Whitney).

$f \in C^1(\mathbb{R}|K) \Leftrightarrow \lim\limits_{x,y\to\xi} \frac{f(x)-f(y)}{x-y} = f'(\xi)$ for all non-isolated $\xi \in K$.

Definition. Every bounded connected component of $\mathbb{R} \setminus K$ is called a gap of K.

Theorem.

$C^1(\mathbb{R}|K) = C^1(K)$ if and only if

$$\lim_{\varepsilon\to 0} \sup \left\{ \frac{\sup\{|x-y| : y \in G\}}{\mathrm{length(G)}} : G \subseteq (x-\varepsilon, x+\varepsilon) \text{ gap of } K \right\} < \infty \text{ for all } x \in K.$$

Examples.

- $C^1(K) \neq C^1(\mathbb{R}|K)$ for the Cantor set.
- For $x_n \searrow 0$ and $K = \{0\} \cup \{x_n : n \in \mathbb{N}\}$ we have $C^1(K) = C^1(\mathbb{R}|K) \Leftrightarrow \limsup\limits_{n\to\infty} \frac{x_n}{x_n - x_{n+1}} < \infty$
- Yes for $x_n = a^{-n}$ with $a > 1$ but No for $x_n = n^p$ with $p < 0$.
- $K = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ and $L = [0,1]$ both satisfy $C^1(M) = C^1(\mathbb{R}^d|M)$ but $K \times L$ does not.